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Lecturer: Bengt E W Nilsson

The Lie algebra $\mathfrak{sl}(3,\mathbb{R})$

Try to construct $\mathfrak{sl}(3,\mathbb{R})$ from two $\mathfrak{sl}(2,\mathbb{R})$ algebras.

$$\text{First one:}\quad \begin{cases} [H^1, E^1_\pm] = \pm\, 2E^1_\pm \\ \big[E^1_+, E^1_-\big] = H^1 \end{cases}$$

Now introduce a second $\mathfrak{sl}(2,\mathbb{R})$. First let $[H^1, H^2] = 0$ since we want representations with two quantum numbers. $\Rightarrow$ The new algebra has rank = 2 if no other commuting operators exist.

The second $\mathfrak{sl}(2,\mathbb{R})$:

$$\begin{cases} [H^2, E^2_\pm] = \pm\, 2E^2_\pm \\ \big[E^2_+, E^2_-\big] = H^2 \end{cases}$$

Now, if all commutators $[\mathfrak{sl}_1(2,\mathbb{R}), \mathfrak{sl}_2(2,\mathbb{R})] = 0$ nothing interesting happens (i.e. we don't get $\mathfrak{sl}(3,\mathbb{R})$): $\mathfrak{g} = \mathfrak{sl}_1(2,\mathbb{R}) \oplus \mathfrak{sl}_2(2,\mathbb{R})$.

So, introduce commutators that mix the two $\mathfrak{sl}(2,\mathbb{R})$'s:

$$\big[H^i, E^i_\pm\big] = \pm\, A^{ji}\, E^i_\pm \quad \text{(no sum over } i\text{)}$$

A^{ji} is not necessarily symmetric, and $A^{11} = A^{22} = 2$ (that's just the two $\mathfrak{sl}(2,\mathbb{R})$ algebras).

To get mixing we need A^{21} and/or $A^{12} \neq 0$.

Recall:

$$\big[E^1_\pm, E^2_\pm\big] = E^\theta_\pm, \quad \big[E^1_\pm, E^2_\mp\big] \text{ will be put to zero.}$$

Note:

$$\big[H^i, E^\theta_\pm\big] = \pm\big(A^{1i} - A^{2i}\big)E^\theta_\pm$$

$$\big[H^i, \big[E^1_\pm, E^2_\mp\big]\big] = \pm\big(A^{1i} - A^{2i}\big)\big[E^1_\pm, E^2_\mp\big]$$

So, both $E^\theta_\pm$ and $\big[E^1_\pm, E^2_\mp\big]$ are possible new step operators in the new algebra. To keep the new algebra as small as possible (this will give $\mathfrak{sl}(3,\mathbb{R})$ in fact) we impose

$$\big[E^1_\pm, E^2_\mp\big] = 0 \quad \text{and} \quad \big[E^i_\pm, E^\theta_\pm\big] = 0. \tag{1}$$

$$\Rightarrow \big[E^i_\mp, E^\theta_\pm\big] = \text{can be computed from the two above.}$$

Now let's compute

$$\big[E^1_+, \big[E^1_-, E^2_+\big]\big] - [E^1_-, \underbrace{\big[E^1_+, E^2_+\big]}_{=E^\theta_+}] = [\text{Jacobi}] = -\big[E^2_+, \big[E^1_+, E^1_-\big]\big] = -\big[E^2_+, H^1\big] =$$

$$= \big[H^1, E^2_+\big] = A^{21} E^2_+$$

Assuming (1) is consistent with $A^{21} \neq 0$. Can we compute A^{21}?

Yes!

$$0=\left[E_-^1,\left[E_+^1,E_+^\theta\right]\right]=[\text{Jacobi},...]\overset{\text{exercise}}{=\!=}-(2A^{21}+A^{11})E_+^\theta.$$

$$\Rightarrow A^{21}=-1.$$

A similar calculation gives $A^{12}=-1$. So

$$A^{ji}=\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

Note: It happens to be symmetric here. It is not a law of nature that this has to be symmetric.

To summarise:

$$A=\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \quad \text{algebra}$$

1) $\left[H^1,H^2\right]=0$
2) $\left[H^i,E_\pm^j\right]=\pm A^{ji}E_\pm^j$
3) six step operators in total $\Rightarrow \dim=8$ (as for $\mathfrak{sl}(3,\mathbb{R})$)
new two: $\left[H^i,E_\pm^\theta\right]=\pm\left(A^{1i}+A^{2i}\right)E_\pm^\theta$

In Fuchs and Schweigert: all commutators are written.

Exercise:

$$\left[E_+^\theta,E_-^\theta\right]=H^1+H^2\equiv H^\theta$$

$\Rightarrow$ a third $\mathfrak{sl}(2,\mathbb{R})$ algebra.

To make this third $\mathfrak{sl}(2,\mathbb{R})$ to appear exactly as the other two we define

$$E_\pm^3:=E_\mp^\theta,\quad H^3=-(H^1+H^2)=-H^\theta$$

Physics: These three $\mathfrak{sl}(2,\mathbb{R})$'s are called, in particle physics, isospin and U and V spin.

This looks like a mess. How can we make sense of all these commutators?

Let the Catan subalgebra consisting of H^1 and H^2 span a vector space: A general element in this vector space is given by coordinates in this basis: $h=h_iH^i$ (compare $\boldsymbol{r}=x_i\boldsymbol{e}^i$ in ordinary geometry). Then:

$$\left[H^i,E^j\right]=\underbrace{\left(\alpha^{(j)}\right)^i}_{\substack{i\text{'th component}\\ \text{of vector}\\ \text{number } (j)\\ =A^{ji}}}\quad E_\pm^j$$

$$\Rightarrow \begin{array}{l} E_+^1:\ \boldsymbol{\alpha}^{(1)}=(2,-1)\\ E_+^2:\ \boldsymbol{\alpha}^{(2)}=(-1,2)\\ E_+^\theta:\ \boldsymbol{\theta}=\boldsymbol{\alpha}^{(1)}+\boldsymbol{\alpha}^{(2)}=(1,1)\end{array}$$

Standard (annoying) notation is to drop the arrow on $\boldsymbol{\alpha}$. (Bengt writes vectors as $\vec{\alpha}$ on the blackboard, rather than using boldface.)

$$\left[h,E_\pm^i\right]=\pm\,\alpha^{(i)}E_\pm^i$$

Let us draw these vectors in the plane.

Figure 1. Using the metric $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

Recalling $E_{\pm}^{3} = E_{\mp}^{\theta}$ (i.e. $\alpha^{(3)} = -\theta$) we must have all three vectors appearing in the same way! What is wrong? The metric used in the diagram is the wrong one! Which metric is the correct one? We need a scalar product between H^1 and H^2. $h \cdot h' = h_i h_j (?)^{ij}$.

Consider the matrix realization of the algebra

$$H^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$E_{+}^{1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{+}^{2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{+}^{\theta} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$E_{-}^{1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{-}^{2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad E_{-}^{\theta} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Then the metric

$$G^{ij} = \mathrm{tr}(H^i H^j) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = A^{ji} \text{ again.}$$

(G^{ij} is a symmetric matrix always, which A^{ji} does not have to be. But they are closely related.)

$$h \cdot h' = h_i G^{ij} h'_j$$

and for the *root vectors* $\alpha^{(i)} = \left\{ \left(\alpha^{(i)} \right)^j \right\}$. Upper index j.

$$\Rightarrow \alpha^{(i)} \cdot \alpha^{(j)} = \left(\alpha^{(i)} \right)^m G_{mn} \left(\alpha^{(j)} \right)^n$$

G_{mn} is the matrix inverse of G^{mn}.

$$G_{mn} = \frac{1}{3} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$\alpha^{(i)} \cdot \alpha^{(j)} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = A^{ij} \text{ again}$$

In the weak interactions one defines an orthogonal basis for the Cartan subalgebra.

$$\tilde{H}^i = \left(\tilde{H}^1, \tilde{H}^2 \right) = \left(\frac{1}{\sqrt{2}} H^1, \frac{1}{\sqrt{6}} (H^1 + 2H^2) \right) = \left(\sqrt{2} H_{I_3}, \sqrt{2} H_Y \right)$$

Isospin and hypercharge. Recall the standard model: $\mathrm{SU}_C(3) \times \mathrm{SU}_W(2) \times \mathrm{U}_Y(1)$.

$$\Rightarrow \tilde{G}^{ij} = \mathrm{tr}\left(\tilde{H}^i \tilde{H}^j \right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So define ${M^i}_j : \tilde{H}^i \equiv {M^i}_j H^j$

$$\Rightarrow {M^i}_j = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{pmatrix}$$

$$\Rightarrow \delta^{ij} = \tilde{G}^{ij} = {M^i}_k {M^j}_l G^{kl} = \left(M G M^T\right)^{ij}$$

and

$$\left[\tilde{H}^i, E_{\pm}^j\right] = {M^i}_k \left(\alpha^{(j)}\right)^k E_{\pm}^j = \left(\tilde{\alpha}^{(j)}\right)^i E_{\pm}^j$$

$\Rightarrow$

$$\begin{aligned} \tilde{\alpha}^{(1)} &= \sqrt{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \tilde{\alpha}^{(2)} &= \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix} \\ \tilde{\theta} = \tilde{\alpha}^{(1)} + \tilde{\alpha}^{(2)} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} \end{aligned}$$

and of course $\tilde{\alpha}^{(i)} \cdot \tilde{\alpha}^{(j)} = A^{ij}$ (basis independent, like $\mathrm{d}s^2 = g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu$).

Figure 2.

Note: This metric can be extended to the whole Lie algebra.

$$\mathcal{H}^{ab} = \mathrm{tr}\left(T^a T^b\right)$$

$\mathcal{H}^{ab}$: Killing form. $T^a = \left(H^1, H^2, E_+^1, E_+^2, E_+^\theta, E_-^1, E_-^2, E_-^\theta\right)$.

$$\Rightarrow \quad \mathcal{H}^{ab} = \left(\begin{array}{cc|ccc|ccc} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right)$$

Diagonalise:

$$\Rightarrow \left(\begin{array}{cccccccc|c} 1 &&&&&&&& \\ & 1 &&&&&&& \\ && 1 &&&&&& \text{non compact} \\ &&& 1 &&&&& \\ &&&& 1 &&&& \\ \hline &&&&& -1 &&& \\ &&&&&& -1 && \text{compact} \\ &&&&&&& -1 & \end{array}\right)$$

Compact SO(3). Maximal compact subalgebra of $\mathfrak{sl}(3, \mathbb{R})$.

The diagonal basis is

$$\tilde{H}^1, \tilde{H}^2, E_+^i + E_-^i, E_+^i - E_-^i$$

$$i = 1 \quad \Rightarrow \quad \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$i = 2 \quad \Rightarrow \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$i = 3 \quad \Rightarrow \quad \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

SO(3) Lie algebra.

$$g = \mathrm{e}^{\alpha(E_+^1 - E_-^1)} = \mathrm{e}^{\alpha \mathrm{i}} \in U(1) \text{ phase, compact,} \quad g = \cos\alpha + (\ ) \sin\alpha$$

$$g = \mathrm{e}^{a(E_+^1 + E_-^1)} = \cosh a + (\ ) \sinh a$$

Are there other rank two Lie algebras we could also have constructed by introducing some step operators?

1) $[E_\mp^1, E_\pm^2] = 0$

2) $[E_\pm^i, E_\pm^\theta] = 0$

In fact there are two other possible Cartan matrices:

$$A = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} \rightarrow B_2 \ (\mathrm{SO}(5))$$

$$A = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} \rightarrow G_2 \text{ (exceptional)}$$

Let us draw all roots in the respective diagrams.

Figure 3.

Representation theory for A_2 $(\mathfrak{sl}(3, \mathbb{C}))$

In physics we classify things using representations of algebras.

$\mathrm{SU}_C(3)$: quarks in 3, antiquarks in $\bar{3}$, gluons in 8.

$\mathrm{SU}_W(3)$: hadrons 8.

$(\mathrm{SU}_W(2) \times U_Y(1))$.

First: Some tensor analysis

SO(3). Invariant tensors δ_{ij} and ε^{ijk}. $\Rightarrow T_i$ in a 3-dimensional representation of SO(3): 3.

Then T_{ij} (3×3). $T_{ij}=T_{[ij]}+T_{(\tilde{ij})}+\delta_{ij}T$. $T_{(\tilde{ij})}$ is traceless. $T_{[ij]}=\varepsilon_{ijk}T_k$: 3.

$$3\otimes 3=3\oplus 5\oplus 1$$

Exercise: Decompose into irreducible representations T_{ijk}.

$\mathfrak{sl}(3,\mathbb{R})$: Invariant tensor is ε^{ijk}.

$$\Rightarrow T_i \text{ in a } 3$$

But T^i is different. T^i is a $\bar{3}$.

$$T_{ij}=T_{[ij]}+T_{(ij)}$$

$$T_{ij}=\varepsilon_{ijk}T^k+T_{(ij)}$$

$$3\otimes 3=\bar{3}+6$$

Also $T_i{}^j$ $(3\otimes\bar{3})$:

$$T_i{}^j=\tilde{T}_i{}^j+\delta_i{}^jT$$

$$3\otimes\bar{3}=8\oplus 1$$

Figure 4.