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Lecturer: Bengt E W Nilsson

Now we are going to shift gear a bit, and go into what is called *representation theory*.

Chapter 5: Representation theory (You can read Fuchs and Schweigert chapters 2 and 3, for the first part of this chapter here.)

Recall the angular momentum algebra in Quantum Mechanics:

$$\boldsymbol{L} = \boldsymbol{q} \times \boldsymbol{p}$$

This is in three dimensions, so

$$L^i = \varepsilon^{ijk} q^j p^k \quad \text{i.e.} \quad L^1 = q^2 p^3 - q^3 p^2$$

In Quantum Mechanics with operators $[\hat{p}^i, \hat{q}^j] = \mathrm{i}\,\delta^{ij}$, where we have put $\hbar = 1$.

$$\Rightarrow \left[\hat{L}^i, \hat{L}^j\right] = \mathrm{i}\,\varepsilon^{ijk}\,\hat{L}^k$$

SO(3) algebra (or SU(2)). The same algebra is obtained for the classical version using the Poisson bracket $[\,\cdot\,,\cdot\,]_{\mathrm{PB}}$ instead of the quantum mechanical commutator.

Note: The Witt algebra is the classical version of the Virasoro algebra. Here, when computing the commutator $[\hat{L}_m, \hat{L}_n]$ we get an answer that needs normal ordering. That leads to a "second" commutator, which leads to the central term or anomaly (or $c/12...$).

Recall also that we have seen at least two other realizations of the angular momentum algebra.

$$\left[T^a, T^b\right] = \mathrm{i}\,\varepsilon^{abc}\,T^c$$

We have seen SO(3) where the $(T^a)^b{}_c = -\,\mathrm{i}\varepsilon^{abc}$ with $a, b, c = 1, 2, 3 = x, y, z$. This is the vector represenation. We have also seen SU(2), where $(T^a)^b{}_c = \frac{1}{2}(\sigma^a)^\beta{}_\gamma$ with $\beta, \gamma = 1, 2$. This is the spinor representation. Having found these representations can we derive or construct all possible representations of this algebra?

(Sakurai $\rightarrow Y_m^{(l)}(\theta, \varphi)$, $l = 1, m = -1, 0, +1$).

To find all representations (finite dimensional) we form

$$L_\pm = L_1 \pm \mathrm{i}\,L_2, \quad L_0 = 2\,L_3$$

$$\Rightarrow \begin{cases} [L_0, L_\pm] = \pm\,2\,L_\pm \\ [L_+, L_- = L_0] \end{cases}$$

$\rightarrow \mathfrak{sl}(2, \mathbb{R})$.

Comment: SO(3), SU(2) and $\mathfrak{sl}(2, \mathbb{R})$ are isomorphic as complex Lie algebras (i.e. as complex vector spaces), but not so if real vector spaces. So as complex vector space we denote this algebra $\mathfrak{sl}(2, \mathbb{C}) \equiv A_1$ in Cartan's classification.

Let us now view L_0, $L_\pm$ as linear operators, i.e. matrices, acting on some d-dimensional vector space (module). Then L_0 will have at least one non-zero eigenvalue.

$$L_0 v_\lambda = \lambda\, v_\lambda$$

where λ is the eigenvalue, or *weight*, and v_λ is the eigenvector, or state.

Example: Spin $\frac{1}{2}$. ($j=\frac{1}{2}$).

$$J^a = \frac{1}{2}\sigma^a, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

So we define new operators J^0, $J_\pm$ as follows:

$$J^0 \equiv \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \Rightarrow \quad J^0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \Rightarrow \quad J^0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = (-1) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Step operators:

$$\begin{cases} J_+ = J_1 + \mathrm{i}\, J_2 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \\ J_- = J_1 - \mathrm{i}\, J_2 = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) \end{cases}$$

$$\begin{cases} J_+ \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = 0 \\ J_+ \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \end{cases}$$

(Stepping up operator.)

We want to generalise this to any possible dimension d ($d=2$ above).

But then $L_\pm v_\lambda$ is also an eigenvector of L_0.

$$L_0(L_\pm v_\lambda) = (\lambda \pm 2)(L_\pm v_\lambda)$$

Check: $L_0(L_+ v_\lambda) = L_+ L_0 v_\lambda + [L_0, L_+] v_\lambda = L_+ \lambda\, v_\lambda + 2 L_+ v_\lambda = (\lambda + 2) L_+ v_\lambda$.

This stepping procedure goes on until we have produced all the d vectors in the d-dimensional vector space (because eigenvectors with different eigenvalues are linearly independent; compare Quantum Mechanics).

Then it must stop at some eigenvalue Λ (λ-max), i.e. $L_+ v_\Lambda = 0$.

Figure 1.

So we suppose it takes N steps to reach the lowest λ-value:

$$(L_-)^N v_\Lambda = v_{\Lambda - 2N}$$

That this coefficient is one is a definition used for the now. If we do $L_- v_{\Lambda-2N} = 0$. $v_{\Lambda-2N}$ is thte lowest weight state, while v_Λ is the heighest weight state.

So defining the vector obtained using L_- with coefficient *one*, $L_- v_\lambda = v_{\lambda-2}$, then we can derive the coefficient obtained using L_+:

$$L_+ v_{\Lambda-2n} = L_+ L_- v_{\Lambda-2n+2} = \underbrace{[L_+, L_-]}_{=L_0} v_{\Lambda-2n+2} + L_- L_+ v_{\Lambda-2n+2} = (L_0 + L_- L_+) v_{\Lambda-2n+2}$$

We define the coefficent r_n by $L_+ v_{\Lambda-2n} = r_n v_{\Lambda-2n+2}$.

$$L_+ v_{\Lambda-2n} = \ldots = (L_0 + L_- L_+) v_{\Lambda-2n+2} = (\Lambda - 2n + 2 + r_{n-1}) v_{\Lambda-2n+2}$$

$$\Rightarrow \quad r_n = r_{n-1} + \Lambda - 2n + 2$$

defined $r_n, n=0,1,2,...$ To solve this we need a starting point. We use that $L_+ v_\Lambda = 0 \Rightarrow r_0 = 0$.

$n=1$: $r_1 = r_0 + \Lambda = \Lambda$.

$$r_n = n(\Lambda - n + 1)$$

Now we have two numbers describing the size of this representation, N and Λ. These can be related as follows:

$$0 = L_+ \underbrace{L_- v_{\Lambda-2N}}_{=0} = (L_0 + L_- L_+) v_{\Lambda-2N} = (r_N + \Lambda - 2N) v_{\Lambda-2N}$$

$$\Rightarrow r_N + \Lambda - 2N = 0$$

$$\Lambda = 2\,N - r_N = 2\,N - N(\Lambda - N + 1)$$

$$\Rightarrow \quad N^2 + (1-\Lambda)N - \Lambda = 0$$

$$\Rightarrow \quad N = -\frac{(1-\Lambda)}{2} \pm \frac{1+\Lambda}{2} = \Lambda \text{ (or } -1)$$

$$N = \Lambda.$$

They are the same thing, and we will not use N any more. The dimension of the representation is $d = N + 1 = \Lambda + 1$.

Heighest weight state v_Λ, lowest weight state $v_{\Lambda-2N} = v_{-\Lambda}$.

Comment: In quantum mechanics we use the spin j as the parameter describing the size: $\lambda = 2\,j$, since $d = 2\,j + 1$, $j = 0, \frac{1}{2}, 1, \frac{3}{2}, ...$ We have states $|j, m\rangle$ with j acting as Λ and m as λ.

m is measured by L_0, and j by the Casimir operator $\boldsymbol{L}^2 = \left(L^1\right)^2 + \left(L^2\right)^2 + \left(L^3\right)^2$ which commutes with L_0 but does *not* belong to the Lie algebra $\mathfrak{so}(3)$ or $\mathfrak{sl}(2,\mathbb{R})$ — it is not linear in L^i.

Recall (Quantum Mechanics) One is always interested in finding the maximal set of commuting operators because they can be diagonalised at the same time.

Example: $d=3$ representation of $\mathfrak{sl}(2,\mathbb{R})$: v_2, v_0, v_{-2}

$$\Rightarrow H = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

(Now we use H instead of L_0). This is the notation for all commuting operators called *Cartan algebra.* Also

$$F = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

(Instead of L_-) which is a stepping down, or lowering operator. Also called negative. $[H, F] = -2F$. Note

$$F^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and the positive or raising operators

$$E = 2\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

(Instead of L_+)

Here

$$E\begin{pmatrix}0\\0\\1\end{pmatrix}=E\,v_{-2}=2\begin{pmatrix}0\\1\\0\end{pmatrix}$$

In quantum mechanics it is standard to divide the r_n coefficients between E's and F's.

$$E=\sqrt{2}\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix},\quad F=\sqrt{2}\begin{pmatrix}0&0&0\\1&0&0\\0&1&0\end{pmatrix}$$

Compare $\mathfrak{sl}(3,\mathbb{R})$. Here we have H^1,H^2,E^1,E^2,F^1,F^2. $[E^1,E^2]=E^\theta\in\mathfrak{sl}(3,\mathbb{R})$.

$$E^1=\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix},\quad E^2=\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix}\quad\Rightarrow\quad E^\theta=\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}\in\mathfrak{sl}(3,\mathbb{R})$$

Exercise: Use $[a^\dagger,a]=1$ to construct the $\mathfrak{sl}(2,\mathbb{R})$ algebra.

The Lie algebra $\mathfrak{sl}(3,\mathbb{R})$

Physics: if we have two particles, like the proton and the neutron, that are rather similar, we should construct a quantum field theory that is almost symmetric under rotations in the two-dimensional complex space $\begin{pmatrix}\psi_n\\\psi_p\end{pmatrix}$. $\Rightarrow$SU(2)-symmetry, which is roughly $\mathfrak{sl}(2,\mathbb{R})$.

But suppose that there are three similar objects, like quarks. Then we need rotations in

$$\begin{pmatrix}\psi_b\\\psi_r\\\psi_g\end{pmatrix}$$

$\Rightarrow$SU(3) or $\sim\mathfrak{sl}(3,\mathbb{R})$.

Try to combine two $\mathfrak{sl}(2,\mathbb{R})$ algebras to get $\mathfrak{sl}(3,\mathbb{R})$.

- First $\mathfrak{sl}(2,\mathbb{R})$:

$$\begin{cases}\left[H^1,E^1_\pm\right]=\pm\,2E^1_\pm\\ \left[E^1_+,E^1_-\right]=H^1\end{cases},\quad E^1_+\equiv E^1, E^1_-\equiv F^1$$

To get a second independent quantum number we need a second Cartan generator H^2

$$\left[H^1,H^2\right]=0$$

i.e. reank = 2. (dimension of the space of commutating operators).

Now, if H^2 commutes with also $E^1_\pm$ and no $E^2_\pm$ exists, then nothing happens, in the sense that the eigenvalue of H^2 is the same for all states.

So to get new, larger representations and algebras we need the full second $\mathfrak{sl}(2,\mathbb{R})$.

$$\begin{cases}\left[H^2,E^2_\pm\right]=\pm\,2\,E^2_\pm\\ \left[E^2_+,E^2_-\right]=H^2\end{cases}$$

One possible case is that $[\mathfrak{sl}_1(2,\mathbb{R}),\mathfrak{sl}_2(2,\mathbb{R})]=0$. Then we get just the sum of two algebras not talking to each other. $\mathfrak{g}=\mathfrak{g}_1\oplus\mathfrak{g}_2$. We have to do something non-trivial, with the E's.

To get something really new we need some non-trivial commutators: mixing $\mathfrak{sl}_1(2,\mathbb{R})$ and $\mathfrak{sl}_2(2, \mathbb{R})$:

$$\left[H^i,E_\pm^j\right]=\pm A^{ji}E_\pm^j \quad \text{(no sum over } j\text{)}$$

with some $A^{ji}\neq 0$ for $i\neq j$. So A^{21} and/or $A^{12}\neq 0$. Note: A^{ji} need not be symmetric. Also note $A^{11}=A^{22}=2$ — that's the two $\mathfrak{sl}(2,\mathbb{R})$'s.

Consider the other possible mixings of the two $\mathfrak{sl}(2,\mathbb{R})$'s:

$$\begin{cases}\left[E_\pm^1,E_\pm^2\right]=E_\pm^\theta\\ \left[E_\pm^1,E_\mp^2\right] \text{“and here I will not write anything”}\end{cases}$$

We can now compute the eigenvalues of these two new operators:

1)

$$\left[H^i,\left[E_\pm^1,E_\pm^2\right]\right]=[\text{Jacobi}]=-\left[E_\pm^1,\left[E_\pm^2,H^i\right]\right]-\left[E_\pm^2,\left[H^i,E_\pm^1\right]\right]=$$

$$=-\left[E_\pm^1,-A^{2i}E_\pm^2\right]-\left[E_\pm^2,\pm A^{1i}E_\pm^1\right]=\pm\left(A^{2i}+A^{1i}\right)\underbrace{\left[E_\pm^1,E_\pm^2\right]}_{\equiv E_\pm^\theta}$$

2) also

$$\left[H^i,\left[E_\pm^1,E_\mp^2\right]\right]=\pm\left(A^{1i}-A^{2i}\right)[\ ,\ ]$$