2009–11–25

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Recall: we constructed $\mathfrak{sl}(3,\mathbb{R})$ from two $\mathfrak{sl}(2,\mathbb{R})$'s.

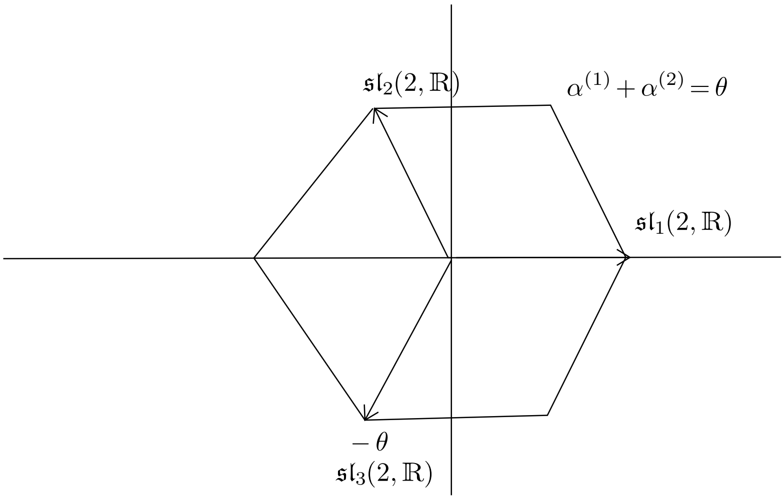

Figure 1. Root diagram

$$E_+^3 \equiv E_-^\theta$$

$$\Rightarrow H_1+H_2+H_3=0$$

- 3 $\mathfrak{sl}(2,\mathbb{R})$ algebras (non-compact).
- one other subalgebra:

$$\underbrace{E_+^1-E_-^1,\quad E_+^2-E_-^2,\quad E_+^3-E_-^3}_{\equiv \mathrm{SO}(3)\,:\ \text{compact}}$$

- Cartan matrix $A^{ij}=\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$: Lie algebra A_2.
- Other rank 2 cases:

$$\begin{cases} A=\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} & \mathrm{SO}(5) \quad B_2 \\ A=\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} & \text{—} \qquad G_2 \text{ (one of the exceptional cases)} \end{cases}$$

These root diagrams have now vectors of different length!

If all root vectors are of the same length the Lie algebra is *simply laced*.

A_2: $\underbrace{\mathfrak{sl}(3,\mathbb{R})}_{\text{split form}}$, $\underbrace{\mathfrak{su}(3),\quad \mathfrak{su}(2,1)}_{\text{compact}}$.

Recall the representation theory of $\mathfrak{sl}(2,\mathbb{R})$

Possible highest weight $\Lambda = 0, 1, 2, \ldots$ $(\Lambda = N)$.

$$[E_+, E_-] = H$$

$$[H, E_\pm] = \pm 2E_\pm$$

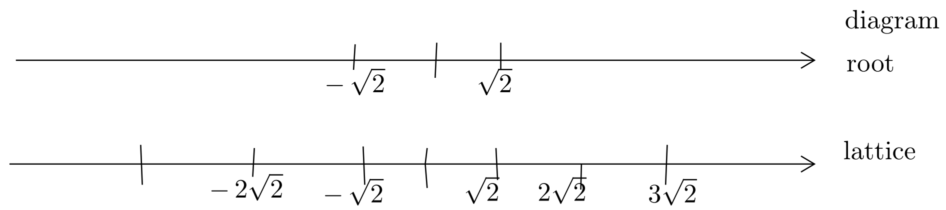

Figure 2.

$$\alpha \cdot \alpha = 2 \quad (= A)$$

$$\Rightarrow \text{Metric on the root lattice} = G^{-1}$$

$$= \frac{1}{2} \quad \left(\alpha G^{-1} \alpha = 2 \cdot \frac{1}{2} \cdot 2 = 2\right)$$

$\Rightarrow$ Dual lattice ($\equiv$ wieght lattice).

Fundamental vector: $\Lambda \cdot \alpha = 1 \Rightarrow \Lambda = \frac{1}{2}$.

$$\Lambda \cdot \Lambda = 2 \quad (\cdot \leftrightarrow a = 2)$$

Lenght of $\Lambda = \frac{1}{\sqrt{2}}$.

For $\mathfrak{sl}(3,\mathbb{R})$: Recall the root lattice; it is generated by

$$\tilde{\alpha}^{(1)} = \sqrt{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \tilde{\alpha}^{(2)} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix}$$

(The components are in the orthonormal coordinate system.)

Find the dual lattice, i.e. the weight lattice: Find $\tilde{\Lambda}_{(i)}$ such that $\tilde{\Lambda}_{(i)} \cdot \tilde{\alpha}^{(j)} = \delta_i{}^j$.

$$\Rightarrow \left\{ \begin{array}{l} \tilde{\Lambda}_{(1)} = \frac{1}{\sqrt{2}}\left(1, \ \frac{1}{\sqrt{3}} \right) \\ \tilde{\Lambda}_{(2)} = \frac{1}{\sqrt{2}}\left(0, \ \frac{2}{\sqrt{3}} \right) \end{array} \right.$$

$$|\tilde{\Lambda}_{(i)}|^2 = \frac{2}{3}$$

(Exercise).

Note: Volume $|\tilde{\alpha}^{(1)} \times \tilde{\alpha}^{(2)}| = \sqrt{3}$. $|\tilde{\Lambda}_{(1)} \times \tilde{\Lambda}_{(2)}| = \frac{1}{\sqrt{3}}$. Ratio $= 3 =$ number of conjugacy classes.

The metric on root lattice

$$\tilde{\alpha}^{(i)} \cdot \tilde{\alpha}^{(j)} = A^{ij} = \left(\begin{array}{cc} 2 & -1 \\ -1 & 2 \end{array} \right)$$

Weight lattice:

$$\tilde{\Lambda}_{(i)} \cdot \tilde{\Lambda}_{(j)} = \left(A^{-1} \right)_{ij} = \frac{1}{3}\left(\begin{array}{cc} 2 & 1 \\ 1 & 2 \end{array} \right)$$

Figure 3.

Define: A highest weight (h.w.) is any point in the weight lattice $= \{n^1 \, \tilde{\Lambda}_{(1)} + n^2 \tilde{\Lambda}_{(2)}, \, n^i \in \mathbb{Z}\}$, with $n^i \geqslant 0$, with the condition that

$$E_+^i v_\Lambda = 0, \quad i = 1, 2.$$

Λ = highest weight, v_Λ = highest weight state, E_+^i = step up operator. ($E_+^\theta = [E_+^1, \, E_+^2]$ is not needed). Also $H v_\Lambda = \Lambda v_\Lambda$. $\Rightarrow$ highest weight representation.

Stepping down: $v_\Lambda \rightarrow E_-^i v_\Lambda =$ new state.

$$H^i \left(E_-^j v_\Lambda \right) \Rightarrow v_{\Lambda' = \Lambda - (\alpha^{(j)})^i}$$

Which other weight vectors (i.e. states) belong to a representation specified by highest weight Λ?

$$v_\Lambda, \quad E_-^i v_\Lambda, \quad E_-^i E_-^j v_\Lambda, \quad \text{etc.}$$

Note: E_+^i not needed to enumerate the states in the representation.

As a first example let $\Lambda = \Lambda_{(1)}$.

$$H^i v_{\Lambda_{(1)}} = \Lambda_{(1)}^i \, v_{\Lambda_{(1)}}$$

$$H^i \left(E_-^j v_{\Lambda_{(i)}} \right) = \left(\left(\Lambda_{(1)} \right)^i - \left(\alpha^{(j)} \right)^i \right) \left(E_-^j v_{\Lambda(1)} \right)$$

$$\Rightarrow \quad E_-^1 \Rightarrow \Lambda_{(1)} \rightarrow \Lambda_{(1)} - \alpha^{(1)}$$

$$E_-^2 \Rightarrow \Lambda_{(1)} \rightarrow \Lambda_{(1)} - \alpha^{(2)}$$

$$\Rightarrow E_-^i E_-^j v_{\Lambda(1)} = v_{\Lambda_{(1)} - \alpha^{(i)} - \alpha^{(j)}}$$

Representation heighest weight (p,q) in the $\Lambda_{(i)}$ basis.

$$\begin{array}{lll} 3: & (1,0) & \Lambda^{\mathrm{hw}} = \Lambda_{(1)} \\ 8: & (1,1) & \theta = \Lambda_{(1)} + \Lambda_{(2)} \\ 6: & (2,0) & \Lambda^{\mathrm{hw}} = 2\Lambda_{(1)} \\ \bar{3}: & (0,1) & \end{array}$$

$$\frac{\text{volume of root lattice}}{\text{volume of weight lattice}} = 3$$

$$w \,\mathrm{mod}\, (\text{root lattice})$$

Example: Do the (2, 0) representation in the vector language.

$$\Lambda^{\mathrm{hw}} = 2\Lambda_{(1)} = (2,0)$$

Step in the $-\alpha^{(1)}$ direction:

$$\alpha^{(1)} = 2\Lambda_{(1)} - \Lambda_{(2)} = (2,-1)$$

$$\Rightarrow \Lambda^{\mathrm{hw}} - \alpha^{(1)} = (0,1)$$

The 1 in $(0,1)$ is positive, step once again, but in $-\alpha^{(2)}$ direction.

$$\Lambda^{\mathrm{hw}} - \alpha^2 = (2,0) - (-1,2) = (3,-2)$$

$2\Lambda_{(1)} - \alpha^{(1)} = \Lambda_{(2)} \rightarrow 2\Lambda_{(1)} - 2\alpha^{(1)} = -2\Lambda_{(1)} + 2\Lambda_{(2)} \rightarrow 2\Lambda_{(1)} - 3\alpha^{(1)} = \text{stops}.$

$\tilde{\alpha}^{(i)}$ orthonormal coordinates, $\alpha^{(i)}$ in the coordinates with metric $A^{-1} = G^{-1}$. But which metric you are using is only important when you write the vector out in a specific basis.

roots: Positive roots: $\alpha > 0$. $\{\alpha^{(1)}, \alpha^{(2)}, \theta\}$ corresponding to the operators with a $+$: $\{E_+^1, E_+^2, E_+^\theta\}$. Negative roots: $\alpha < 0$: the rest.

simple roots: as many as the dimension of the root diagram such that all other positive roots have positive integer coordinates in this basis.

Multiplicity of the root spaces (space of operators with weight vector λ)

Can be computed using Freudenthal recursion relation.