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Example: Elementary particle physics.

The Standard Model (SM) of elementary particle physics and the Minimal Supersymmetric Standard Model (MSSM).

Lepton. $G_{\mathrm{SM}} = U_Y(1) \times \mathrm{SU}_{\mathrm{w}}(2) \times \mathrm{SU}_{\mathrm{c}}(3)$ – hypercharge, weak and colour

$\mathrm{SU}(2) \times \mathrm{SU}(3)$ singlets $(e_{\mathrm{R}})_Y$ — non-zero hypercharge, $(u_{\mathrm{R}})_Y, (d_{\mathrm{R}})_Y$, etc for the *families*.

SU(2) doublet: $\begin{pmatrix} e \\ \nu \end{pmatrix}_{\mathrm{L},\, Y}$. L is left-handed, R is right-handed. SU(3) singlet.

SU(2) doublets: $\begin{pmatrix} u \\ d \end{pmatrix}_{\mathrm{L},\, Y}$, SU(3) triplets.

In general we have fields with many indices.

Example: The up quark:

$$u_{\alpha b} \to \mathrm{e}^{\mathrm{i}\alpha Y} (u_2)_{\alpha}{}^{\beta} (u_3)_a{}^b u_{\beta b}$$

$u_2 \in \mathrm{SU}(2), u_3 \in \mathrm{SU}(3)$. Plus a hidden index for the Lorentz representation.

Note: In general it is very important to be able to “multiply” representations: this we will do later. Recall, in SU(2): $\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$.

Note: Matrices $\in G$ are in some representation of the group, but the fields correspond to the vector space these matrices act on, called *modules*. In phyisics we refer to both representations and modules as representations.

We will now define general classes of groups by generalising SU(2) and SO(3) matrices: these are called *matrix groups*. They provide almost all finite dimensional groups; the other ones (see later) are called *exceptional groups*. (Read FS section 9.7.)

We start by considering completely general $n \times n$ matrices.

$$A = \begin{pmatrix} a_1{}^1 & a_1{}^2 & \cdots & a_1{}^n \\ a_2{}^1 & & & \\ \vdots & & & \\ a_n{}^1 & \cdots & & a_n{}^n \end{pmatrix}, \quad \text{with } a_i{}^j \in \mathbb{R}$$

Using matrix multiplication $A \cdot B = C$ reads

$$(A \cdot B)_i{}^j = \sum_k a_i{}^k \, b_k{}^j = c_i{}^j$$

we see that we get a group from all such matrices provided $\det A \neq 0$. So $A \in \mathrm{GL}(n, \mathbb{R})$ if $\det A \neq 0$.

Recall

$$\det A = \sum_{\substack{i_1, \ldots, i_n \\ j_1, \ldots, j_n}} \frac{1}{n!} \varepsilon^{i_1 \ldots i_n} a_{i_1}{}^{j_1} \cdots a_{i_n}{}^{j_n} \, \varepsilon_{j_1 \ldots j_n}$$

$\varepsilon^{i_1 \cdots i_n}$ (and $\varepsilon_{i_1 \ldots i_n}$) are totally antisymmetric with $\varepsilon^{12 \cdots n} = +1$. Normally we simplify this a bit, doing the sum over j:

$$\det A = \sum_{i_1 \cdots i_n} \varepsilon^{i_1 \cdots i_n} a_{i_1}{}^1 \, a_{i_2}{}^2 \cdots a_{i_n}{}^n$$

Exercise: Write out $\det A$ for $n=5$.

Then $\det A \neq 0 \Rightarrow A^{-1}$ exists, since

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} \cdots & \cdots \\ \vdots & \ddots \end{pmatrix}$$

Now we can use instead other kinds of "numbers".ng For instance let $z_i{}^j \in \mathbb{C}$.

$$A = \begin{pmatrix} z_1{}^1 & \cdots & z_1{}^n \\ \vdots & \ddots & \vdots \\ z_n{}^1 & \cdots & z_n{}^n \end{pmatrix}$$

In fact we can also use (FS p. 48), $\mathbb{H}$ (quaternions), or some finite field (in the mathematical sense) like $\mathbb{Z}_p$.

$\mathbb{H}$ has three units i, j, k with $i^2 = j^2 = k^2 = -1, i j = k$. In effect $(i, j, k) = \mathrm{i}\,\sigma^i$ (Pauli matrices).

$$q \in \mathbb{H} \quad q = a + b\,i + c\,j + d\,k, \quad a, b, c, d \in \mathbb{R}$$

We get groups like $\mathrm{GL}(n, \mathbb{H})$, and $\mathrm{GL}(n, \mathbb{Z}_p)$. $\mathbb{Z}_p = \{0, 1, 2, \ldots, p-1\}$, $\mathbb{Z} \bmod p$.

Read FS page 48, about rings and fields.

$\mathbb{Z}$ s not a field, and thus not good in $\mathrm{GL}(n, \mathbb{Z})$.

We get *special linear groups* by setting $\det A = 1$.

$$\Rightarrow \mathrm{SL}(n, \mathbb{R}), \quad \mathrm{SL}(n, \mathbb{C}), \quad \mathrm{SL}(n, \mathbb{H})$$

(Can't use octonions, since groups are defined to be associative.)

Division algebras (FS page 71–73): $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. In $\mathbb{O}$ we have seven complex units, $e_1, \ldots, e_7$.

Note: There are two other sub-Lie groups that can easily be identified. $\mathrm{sol}(n)$ (solvable). A solvable matrix is a special matrix ton the form

$$S = \begin{pmatrix} s_1{}^1 & s_1{}^2 & \cdots & s_n{}^n \\ 0 & s_2{}^2 & & \neq 0 \\ 0 & 0 & \ddots & \\ 0 & 0 & 0 & s_n{}^n \end{pmatrix}$$

$\mathrm{nil}(n)$: nilpotent:

$$n = \begin{pmatrix} 1 & & & & \\ 0 & 1 & & \neq 0 & \\ 0 & 0 & 1 & & \\ 0 & 0 & 0 & \ddots & \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Physics:

$$n_1 = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}, \quad n_2 = \begin{pmatrix} 1 & \alpha & \beta \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow n_3 = n_1 n_2 = \begin{pmatrix} 1 & \alpha + a & \beta + b + a\gamma \\ 0 & 1 & \gamma + c \\ 0 & 0 & 1 \end{pmatrix}$$

This example is called the Heisenberg group and is related to the harmonic oscillator operators $a, a^\dagger, \hbar$ and to $p, q, \hbar$.

Another set of subgroups of $\mathrm{GL}(n, \mathbb{F})$ for some field $\mathbb{F}$ $(\mathbb{R}, \mathbb{C}, \mathbb{H}, ...)$

Orthogonal groups $\mathrm{O}(n, \mathbb{F})$.

$$\mathrm{O}(n, \mathbb{R}): \quad R \in \mathrm{GL}(n, \mathbb{R}) \ \text{ such that } G = \mathbf{1} \text{ is invariant, i.e. } R^T G R = G \ \Rightarrow \ R^T R = \mathbf{1}$$

$$O(p, q; \mathbb{R}): \quad G = \begin{pmatrix} \mathbf{1}_p & 0 \\ 0 & -\mathbf{1}_q \end{pmatrix} \equiv \mathbf{1}_{(p,q)}, \quad R^T G R = G$$

$\mathrm{U}(n) \subset \mathrm{GL}(n, \mathbb{C})$ such that $U^\dagger G U = G, G = \mathbf{1}$. $\mathrm{U}(p, q)$ has $G = \mathbf{1}_{(p,q)}$.

Symplectic: $\mathrm{Sp}(2n, \mathbb{R})$, $\mathrm{Sp}(2n, \mathbb{C})$

$$M \in \mathrm{Sp}(2n, \mathbb{R}) \ \text{ if } \ M^T J M = J \ \text{ where } \ J = \begin{pmatrix} 0 & \mathbf{1}_{n \times n} \\ -\mathbf{1}_{n \times n} & 0 \end{pmatrix}$$

Note: $\mathrm{Sp}(2n, \mathbb{R})$ can be related to $\mathrm{GL}(n, \mathbb{H})$.

Next: we note a theorem.

Theorem: $H_1 \subset G, H_2 \subset G$ then also $H_1 \cap H_2$ is a subgroup of G.

Then we define groups like

$$\mathrm{SO}(p, q) = \mathrm{O}(p, q) \cap \mathrm{SL}(p + q, \mathbb{R})$$

$\mathrm{SO}(p, q)$ has both conditions: $A^T \mathbf{1}_{(p,q)} A = \mathbf{1}_{(p,q)}$ and $\det A = 1$.

$$\mathrm{SU}(p, q) = U(p, q) \cap \mathrm{SL}(p + q, \mathbb{C})$$

Example: $\mathrm{SU}(1, 1)$. $U \in \mathrm{SL}(2, \mathbb{C})$ if

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{with } a, b, c, d \in \mathbb{C}, \quad a d - b c = 1$$

and

$$U^\dagger \mathbf{1}_{(1,1)} U = \mathbf{1}_{(1,1)}$$

$$\begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} |a|^2 - |c|^2 & a^* b - c^* d \\ ... & ... \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\Rightarrow \quad U = \begin{pmatrix} a & b \\ b^* & a^* \end{pmatrix}$$

with $|a|^2 + |b|^2 = 1 \sim$ hyperbolic space. Exercise: Show this.

(Recall $\mathrm{SU}(2)$: $U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$, $|a|^2 + |b|^2 \sim S^3$.

Comments: The dimension of these groups can be determined: dim = number of real parameters.

$$\begin{array}{ll}
\mathrm{GL}(n,\mathbb{R}) & \text{number} = n^2 \\
\mathrm{SL}(n,\mathbb{R}) & \text{number} = n^2-1 \\
\mathrm{GL}(n,\mathbb{C}) & \text{number} = 2n^2 \\
\mathrm{SL}(n,\mathbb{C}) & \text{number} = 2n^2-2 \\
\mathrm{SO}(n) & \text{number} = n^2 - \dfrac{n(n+1)}{2} = \dfrac{n(n-1)}{2} \\
\mathrm{SU}(n) & \text{number} = 2n^2 \underbrace{-n^2}_{\substack{\text{from}\\ U^\dagger U=1}} \underbrace{-1}_{\text{det}} = n^2-1 \\
\mathrm{SU}(2) & \text{number} = 3 \text{ (quantum mechanics)} \\
\mathrm{SU}(3) & \text{number} = 8 \text{ (gluons in the Standard Model)} \\
\mathrm{SL}(2,\mathbb{C}) & \text{number} = 6 \text{ (is actually related to SO(1,3))}
\end{array}$$

Example: $\mathrm{GL}(n,\mathbb{C})$ has an invariant subgroup, namely $\mathrm{SL}(n,\mathbb{C})$. Exercise: show this.

Note: The concepts from finite group theory appear also in Lie group theory:

Definition: subgroup $H \subset G$.

Definition: proper subgroup $H \neq \{e\}, H \neq G$.

Definition: Invariant subgroup. $h \in H, g \in G \Rightarrow g\,h\,g^{-1} \in H$, for all h, g.

Definition: G is simple if it has no proper invariant subgroups.

Definition: G is semi-simple if it has no *abelian* proper invariant subgroup.

We will later classify all Lie groups by classifying their Lie algebras. (next time)

Example: $\mathrm{SU}(n), \mathrm{SO}(n), \mathrm{Sp}(n,\mathbb{R}), \mathrm{SL}(n,\mathbb{R})$ are all simple, except $\mathrm{SO}(4)$.

- Topological properties.

Definition: A Lie group G is *connected* if all elements $g \in G$ are continuously related. Example: SO(3) is connected, but O(3) is not connected, because det $= +1$ and det $= -1$ are not related — no continuous way to transform the one into the other.

Example: U(2) and $\mathrm{SU}(2) \sim S^3$ are both connected.

Definition: A Lie group is *compact* if all elements in the matrix take values in finite ranges.

Example: U(1): $\quad U^\dagger U = 1 \Rightarrow |a_i{}^j| \leqslant 1$. Exercise: Show this.

Example: $\mathrm{SO}(1,3)$ is not compact.

$$\Lambda_\mu{}^\nu = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \det\Lambda = \gamma^2(1-\beta^2) = 1$$

and here

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}, \quad \beta = \frac{v}{c}, \quad -c \leqslant v \leqslant c$$

Example: $\mathrm{SU}(1,1)$

$$U = \begin{pmatrix} a & b \\ b^* & a^* \end{pmatrix}$$

$|a|^2 - |b|^2 = 1$ not compact.

Example: $\mathrm{SL}(2,\mathbb{R})$

$$A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad-bc=1$$

not compact.

• Cosets

If $H\subset G$, then G/H is a "coset space".

Example: $\mathrm{SO}(8)/\mathrm{SO}(7)\approx S^7$ (7-dimensional sphere). $\mathrm{SO}(n)/\mathrm{SO}(n-1)\approx S^n$. These S^n are not groups, in general.

If H is an invariant (normal) subgroup then $G/H=\text{group}$. Example:

$$\mathrm{SO}(4)/\mathrm{SO}(3)\approx S^3\approx \mathrm{SU}(2)$$

Comments: $\mathrm{SO}(8)/\mathrm{SO}(7)\sim S^7$ is not a group, but almost. (This has to do with octonions. S^7 is a very curios object, very close to being a group.)

$$o=a_0+a_1e_1+\cdots+a_7e_7\colon\quad |o|^2=1\Leftrightarrow S^7$$

$$\mathbb{C}\colon\quad |z|^2=1\quad\Leftrightarrow\quad S^1\sim \mathrm{U}(1)$$

$$\mathbb{H}\colon\quad \sum_{i=0}^{3}|e_i|^2=1\quad\Leftrightarrow\quad S^3\sim \mathrm{SU}(2)$$

These coset spaces are crucial in Kaluza–Klein theories where one compactifies sometimes on coset spaces.

Example: In supergravity in 11 dimensions: One can compactify

$$M_{11}=\underbrace{M_4}_{\substack{\text{space-}\\\text{time}}}\times S^7$$

$\Rightarrow$ a spacetime theory with gravity and an SO(8) gauge theory.

Centre

Recall the relation between $\mathrm{SU}(2)\sim S^3$ and $\mathrm{SO}(3)\sim\mathbb{R}P^3$. This is related to the centre $\mathbb{Z}_2$ of SU(2). In general $\mathrm{SU}(N)$ has a number of elements $g_n\in\mathrm{SU}(N)$ that commute with *all* $g\in\mathrm{SU}(N)$.

$$g_n=\mathrm{e}^{2\pi\mathrm{i}\frac{n}{N}}\mathbf{1},\quad n=0,1,\ldots,N-1$$

This set of elements is called the centre $\mathbb{Z}_N=\{g_n\}$ of the Lie group $\mathrm{SU}(N)$.

Note: $\mathrm{SU}(N)/\mathbb{Z}_n$ is still a group since $\mathbb{Z}_N$ is an invariant subgroup.

Example: $\mathrm{SU}(2)/\mathbb{Z}_2\approx\mathrm{SO}(3)$.

Figure 1.

Dividing out $\mathbb{Z}_n$ turns the *simply connected* $\mathrm{SU}(N)$ into a *multiply connected* subgroup.

The simply connected group is called the *universal covering group* of the others.

Note: Groups with different names can (in low dimensions) be the same!

Example: $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{Sp}(2,\mathbb{R})$:

$$A\in \mathrm{SL}(2,\mathbb{R}): \quad A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad-bc=1$$

$$A\in \mathrm{Sp}(2,\mathbb{R}): \quad A^T J A^T = J, \quad J=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 & ad-bc \\ bc-ad & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\Rightarrow ad-bc=1$$

These were equivalent groups.

Example: Centre constructions

$\mathrm{SO}(N)$ has non-trivial cycles ($\pi_1(\mathrm{SO}(N))=\mathbb{Z}_2$). Introduce Spin(N) as

$$\mathrm{Spin}(N)/\mathbb{Z}_2 = \mathrm{SO}(N)$$

Example: $N=3$: Spin(3)=SU(2).

$N=4$: $\quad \mathrm{Spin}(4)=\mathrm{SU}(2)\times\mathrm{SU}(2)$.

$N=5$: $\quad \mathrm{Spin}(5)=\mathrm{Sp}(4,\mathbb{R})$

$N=6$: $\quad \mathrm{Spin}(6)=\mathrm{SU}(4)$.

All $\mathrm{Spin}(N)$ for $N>6$ are not related to other groups.