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Lecturer: Bengt E W Nilsson

Recall: The Great Orthogonality Theorem.

$$\sum_{R \in G} \left(\Gamma^{(i)}(R)\right)^*_{\mu\nu} \left(\Gamma^{(j)}(R)\right)_{\alpha\beta} = \frac{g}{l_i}\, \delta_{ij}\, \delta_{\mu\alpha}\, \delta_{\nu\beta}$$

with $g = |G|$ being the order of the group G and l_i being the dimension of the representation (i). Only irreducible representations appear as $\Gamma^{(i)}$. Summing over matrix elements we can get relations for characters. ($\chi(R) \equiv \mathrm{Tr}\left(\Gamma^{(i)}(R)\right)$). $\Rightarrow$ Orthogonality relations for rows and columns of the character table.

Summary of rules leading to character tables

(i) The number of irreducible representations = the number of classes.

(ii) $\sum_i (l_i)^2 = g$ where we sum over all irreducible representations of G.

(iii) The orthogonality of the character table columns

(iv) The orthogonality of the character table rows

(v) $N_j \chi^{(i)}(\mathcal{C}_j)\, N_k \chi^{(i)}(\mathcal{C}_k) = l_i \sum_l c_{jkl}\, N_l\, \chi^{(i)}(\mathcal{C}_l)$ with c_{jkl} being class multiplication constants.

Exercise: Use (i) – (iv) to derive the D_3 character table directly and check property (v).

Another use is to decompose *reducible* representations into irreducible representations. For any representation $\Gamma(R)$ with character $\chi(R)$ we have

$$\chi(R) = \sum_i a_i\, \chi^{(i)}(R) \quad \forall R \in G$$

(sum over irreducible representations)

$$\Rightarrow \quad a_i = \frac{1}{g} \sum_k N_k \left(\chi^{(i)}(\mathcal{C}_k)\right)^* \chi(\mathcal{C}_k)$$

Exercise: Show this.

Exercise: Find all irreducible representations of all finite groups of prime order. $\Rightarrow$ abelian $\Rightarrow$ all elements are a separate class $\Rightarrow$ the number of irreducible representation $= |G|$ $\Rightarrow$ from

$$\sum_{i=1}^{|G|} (l_i)^2 = |G|$$

we see that all irreducible representations are one-dimensional, i.e. complex numbers solving $(R)^{|G|} = E$.

$$\Gamma^{(r)} = \mathrm{e}^{2\pi \mathrm{i} r/|G|}, \quad r = 0, \dots, |G| - 1$$

Now consider the following funny representation called the *regular representation*: Write the multiplication table for D_3 as

	E	A	B	C	D	F
E	E	A	B	C	D	F
A^{-1}	A	E				
B^{-1}			E			A
C^{-1}				E	A	
D^{-1}				A	E	
F^{-1}			A			E

This gives you a six-dimensional representation:

$$\Gamma^{\text{reg}}(E) = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}$$

$$\Gamma^{\text{reg}}(A) = \begin{pmatrix} & 1 & & & & \\ 1 & & & & & \\ & & & & & 1 \\ & & & & 1 & \\ & & & 1 & & \\ & & 1 & & & \end{pmatrix}$$

$$\Rightarrow \begin{cases} \chi(E) = 6 = \chi(\mathcal{C}_1) \\ \chi(\mathcal{C}_2) = 0 \\ \chi(\mathcal{C}_3) = 0 \end{cases}$$

$\Rightarrow$ Theorem (the Celebrated Theorem) $a_j = l_j$

$$\chi^{(\text{reg})} = \sum_i l_j \chi^{(j)}$$

Proof.

$$a_j = \frac{1}{g} \sum_R \left(\chi^{(j)}(R)\right)^* \underbrace{\chi^{(\text{reg})}(R)}_{\to \chi^{(\text{reg})}(E) = g} = \frac{1}{g}\left(\chi^{(j)}(E)\right)^* g = \left(\chi^{(j)}(E)\right)^* = l_j$$

$$\Rightarrow \quad \sum_i (l_i)^2 = g$$

(from before we only had $\leqslant$, now we have $=$.)

Equality follows since

$$\Gamma^{\text{reg}} = \begin{pmatrix} \Gamma^{(1)} & & & & & & & \\ & \Gamma^{(1)} & & & & & & \\ & & \ddots & & \left| l_1 \,\text{times} \right. & & & \\ & & & \Gamma^{(1)} & & & & \\ & & & & \Gamma^{(2)} & & & \\ & & & & & \Gamma^{(2)} & \left| l_2 \,\text{times} \right. & \\ & & & & & & \ddots & \\ & & & & & & & \ddots \end{pmatrix}$$

Size lhs = g, size rhs $= \sum_i (l_i)^2$. $\square$

We will now briefly discuss two kinds of important applications:

1) Basic Galois theory.

2) Quantum Mechanics.

1) Galois theory.

Consider an n-th order polynominal equation in $z\in\mathbb{C}$ with roots $z_i, i=1,\ldots,n$.

$$(z-z_1)(z-z_2)\cdots(z-z_n)=0$$

$$z^n-I_1z^{n-1}+I_2z^{n-2}+\cdots+(-1)^nI_n=0$$

which we aim to solve in general. Here

$$I_1=\sum_{i=1}^{n} z_i,\quad I_2=\sum_{i<j} z_iz_j,\quad \ldots\quad I_n=z_1\,z_2\cdots z_n$$

These I_i's are all invariant under permutation of z_i's, i.e. under the *Galois group* S_n.

Note now that any function invariant under S_n is expressible as

$$f^{\text{inv}}=f(I_1,I_2,\ldots,I_n)$$

Problem: Find the z_i's expressed in terms of the I_i's. (This is equivalent to solving the n-th order equation, not knowing the z_i's.)

Galois theorem: A solution $z_i\in\mathbb{C}$ exists if and only if there is a chain of subgroup in the subgroup diagram of S_n such that each arrow in the chain connects a G and an H satisfying (i) H is an invariant subgroup of G and (ii) G/H is abelian.

Example of subgroup diagram:

$$\begin{array}{ccccc} & & S_3 & & \\ & \swarrow & & \searrow & \\ A_3 & & & & S_2 \\ & \searrow & & \swarrow & \\ & & E & & \end{array}$$

The theorem also says this is constructive.

$\bullet S_3$.

$$\begin{array}{ll} S_3/A_3=S_2 & \text{abelian} \\ A_3/I=A_3 & \text{abelian} \end{array}$$

$\Rightarrow$ all cubic equations can be solved. Home problem: Quartic equation.

Note: From 5'th order and up the genereal equation cannot be solved.

Exercise: Solve the cubic equation. Is this possible in general? Yes according to Galois' theorem.

$$\begin{array}{ccccc} & & S^3 & & \\ & \swarrow & & \searrow & \\ A_3 & & & & S_2 \\ & \searrow & & \swarrow & \\ & & E & & \end{array}$$

Red arrows: satisfy Gallois condition.

Exercise: Does the other chain (via S_2) also work? I.e. is S_3/S_2 a group?

$S_3/A_3=S_2$ is abelian, and $A_3/E=A_3$ is also abelian.

Solution: $z^3 - I_1 z^2 + I_2 z - I_3 = 0$. Roots r_1, r_2, r_3.

$$\begin{cases} I_1 = r_1 + r_2 + r_3 \\ I_2 = r_1 r_2 + r_2 r_3 + r_1 r_3 \\ I_3 = r_1 r_2 r_3 \end{cases}$$

These are all invariant under S_3 $(= D_3)$. First we need $A_3/E = A_3$ and its character table.

$$A_3 = \left\{ E, \underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}}_{\equiv (123)}, \underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}}_{\equiv (132)} \right\}$$

Character table

	I	$\{(123)\}$	$\{(132)\}$
$\Gamma^{(1)}$	1	1	1
$\Gamma^{(2)}$	1	ω	ω^2
$\Gamma^{(3)}$	1	ω^2	ω

where $\omega^3 = 1$, i.e. $\omega = \mathrm{e}^{2\pi \mathrm{i}/3}$. We need to represent these one-dimensional representations on the roots r_1, r_2 and r_3.

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = \begin{pmatrix} r_1 + r_2 + r_3 = I_1 \\ r_1 + \omega r_2 + \omega^2 r_3 \\ r_1 + \omega^2 r_2 + \omega r_3 \end{pmatrix}$$

These are three functions of the roots.

That is:

$$\Gamma^{(1)}: \quad v_1 = r_1 + r_2 + r_3 = I_1$$

so v_1 is invariant under A_3 (in fact under the whole of S_3).

$\Gamma^{(2)}$: v_2 is invariant under I and goes to ωv_2 under $(12\,3)$ and to $\omega^2 v_2$ under (132).

$\Gamma^{(3)}$: v_3 similar.

Check:

$$(123) v_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} v_2 = r_2 + \omega r_3 + \omega^2 r_1 = \omega^2 v_2?$$

not correct. We should have used $(123)^{-1}$ instead.

$$(123)^{-1} v_2 = (132)\, v_2 = \omega\, v_2$$

Why is this?

Answer: Follow Wigner: Define an operator $\mathcal{O}_A \in G$ acting on functions of r_i as follows

$$\mathcal{O}_A f(A(r_i)) = f(r_i) \quad \Rightarrow \quad \mathcal{O}_A\, f(r_i) = \underbrace{f(A^{-1}(r_i))}_{\text{what we did above}}$$

So we have now three functions of r_i: $v_1 = I_1$ (S_3 invariant) while v_2 and v_3 are not invariant under A_3.

So, next step. We need functions invariant under A_3 but not under $S_2 = S_3/A_3$. (1st step A_3/I). These are $(v_2)^3$ and $(v_3)^3$: these are invariant under A_3 (trivially, $\omega^3 = 1$) but not invariant under S_2.

Check

$$(12)(v_2)^3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}(v_2)^3 = \left(\underset{2}{r_{\cancel{1}}} + \omega \underset{1}{r_{\cancel{2}}} + \omega^2 r_3\right)^3 = \omega^3 (v_3)^2 = (v_3)^3$$

It takes $(v_2)^3 \to (v_3)^3$.

Since S_2 is the Galois group of a second order equation we have

$$\left(x - (v_2)^3\right)\left(x - (v_3)^3\right) = 0$$

$$\Rightarrow \begin{cases} J_1 = (v_2)^3 + (v_3)^3 \\ J_2 = (v_2)^3 \cdot (v_3)^3 \end{cases}$$

which are the two combinations that are invariant under S_2. $\Rightarrow J_1$ and J_2 are S_3 invariant.

$$\Rightarrow J_1 = \sum_{i+2j+3k=3} A_{ijk} I_1^i I_2^j I_3^k$$

$$\Rightarrow J_2 = \sum_{i+2j+3k=6} B_{ijk} I_1^i I_2^j I_3^k$$

Then we see that:

$$\left\{ \begin{array}{c} (v_2)^3 \\ (v_3)^3 \end{array} \right\} = \frac{1}{2} J_1 \pm \sqrt{J_1^2 - J_2}$$

$$\Rightarrow \quad \left\{ \begin{array}{c} v_2 \\ v_3 \end{array} \right\} = \left(\frac{1}{2} J_1 \pm \sqrt{}\right)^{1/3}$$

Finally the solution is obtained by inverting the character table

$$\begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = \frac{1}{3} \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}}_{=(\text{character table})^{-1}} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

Trick: Procedure much simpler after a Tschirnhaus transformation. $z = y + \frac{1}{3} I_1$. $\Rightarrow$ The new $I_1 = 0$. *Exercise:* show this.

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2. Quantum Mechanics (Tinkham)

In Quantum Mechanics we define the physics of a system by its Hamiltonian (or energy):

Example: The hydrogen atom: H for the e^- is:

$$H = \frac{p^2}{2m} + V(r)$$

with $r =$ distance to the proton.

$$V(r) = -\frac{e^2}{r}$$

$\Rightarrow$ Schrödinger equation: $\hat{H}\psi_n = E_n\psi_n$

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + V(r)$$

Here E_n are the possible energy eigenvalues. The symmetry group of this equation is SO(3), rotations in three dimensions.

To get back to D_3 we can instead of proton consider three protons at the corners of a triangle. V is complicated now. But the Schrödinger equation is still invariant under D_3.

So let $\hat{P}_R$ be an operator corresponding to an element in D_3, then

$$\hat{P}_R\hat{H} = \hat{H}\hat{P}_R$$

or

$$\hat{H}\left(\hat{P}_R\psi_n\right) = E_n\left(\hat{P}_R\psi_n\right)$$

$\Rightarrow \hat{P}_R\psi_n$ has the same eigenvalue as ψ_n itself.

Final comment

1) If $G/H_1 = H_2$ and $G/H_2 = H_1$, then $G = H_1 \times H_2$ (direct product). This means that all irreducible representations of G are tensor products of H_1 and H_2.

$$(\Gamma(A))_{AB} = (\Gamma(B))_{ab}(\Gamma(C))_{\alpha\beta}$$

$$(\Gamma(G))_{AB} = (\Gamma(H_1))_{ab}(\Gamma(H_2))_{\alpha\beta}$$

2) If $G/H_1 = H_2$ and $G/H_2 \neq H_1$. H_1 is one invariant subgroup. H_2 is not an invariant subgroup. $G = H_2 \ltimes H_1$. Semi-direct product.

Exercise: Poincaré group. Which part is invariant subgroup?