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Chapter 2: Representations & characters

Definition: *Matrices* satisfying the multiplication table of a group G are called a *representation*, often denoted $\Gamma(R)$ for $R\in G$.

Example: The group $\{1,-1\}$ is a representation of D_3:

$$\begin{array}{rcl} E,D,F & \rightarrow & 1 \\ A,B,C & \rightarrow & -1 \end{array}$$

This is an *unfaithful* representation (i.e., not one-to-one).

Definition: Satisfying the multiplication table means

$$G\ni A_i \quad\rightarrow\quad \Gamma(A_i) \quad \text{then} \quad A_iA_j=A_k \quad\Rightarrow\quad \Gamma(A_i)\Gamma(A_j)=\Gamma(A_k)$$

Definition: The *dimension of the representation* = the size of the matrices!

Note: A *similarity transformation* just changes the basis used and thus

$$\begin{cases} \Gamma_i'=s^{-1}\Gamma_i s \\ \Gamma_i\Gamma_j=\Gamma_k \end{cases}$$

$$\Rightarrow\quad \Gamma_i'\Gamma_j'=s^{-1}\Gamma_i s\, s^{-1}\Gamma_j s=s^{-1}\Gamma_i\Gamma_j\, s= s^{-1}\Gamma_k s=\Gamma_k'$$

Matrices related by similarity transformations are called *equivalent representations*.

Note: If $\Gamma^{(1)}$ and $\Gamma^{(2)}$ are two (in)equivalent representations, then

$$\Gamma=\begin{pmatrix} \Gamma^{(1)} & 0 \\ 0 & \Gamma^{(2)} \end{pmatrix}$$

is also a representation (non-equivalent to $\Gamma^{(1)}$ and $\Gamma^{(2)}$). But one might get this representation in a form (after an arbitrary similarity transformation) where this block structure is not obvious.

Definition: If a representation can be put in block form by a suitable choice of basis then the representation is *reducible*, otherwise *irreducible*.

Question: How can we tell if a representation is irreducible or not?

To find the similarity transformation might be difficult in general (if it exists).

However, the problem will be rather easy if we can find *all irreducible representations*.

First: Can we express information about a representation that is independent of basis (i.e., is the same for all equivalent representations)?

The answer is the *character*.

Recall examples of representations of D_3:

1. $\forall A_i\rightarrow 1$

2. $E,D,F\rightarrow 1.\quad A,B,C\rightarrow -1.$

3. 2-dimensional matrix representation given in a previous lecture (faithful).

In fact there is another very natural representation of $D_3 = S_3$ (permutation group) in terms of 3×3 matrices.

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 \rightarrow 2 \\ 2 \rightarrow 1 \\ 3 \rightarrow 3 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Note: The alternative group $A_3 \subset D_3$ is defined as the set of 3×3 matrices of determinant $= +1$.

Definition: Character of representation $\Gamma^{(i)}$:

$$\chi^{(i)} = \{\chi^i(E), \chi^{(i)}(A_{\mathfrak{r}}), \ldots, \chi^{(i)}(A_g)\}$$

($g = |G|$) where each

$$\chi^{(i)}(A_j) = \mathrm{Tr}\,\Gamma^{(i)}(A_j) = \sum_{\mu=1}^{l_i} \left(\Gamma^{(i)}(A_j)\right)_{\mu\mu}$$

where l_i is the dimension of the representation.

Note: $\mathrm{Tr}(s^{-1}\Gamma s) = \mathrm{Tr}(\Gamma)$.

Consider now the *character table* from the previous representation (1), (2), (3).

	$\mathcal{C}_1$	$3\mathcal{C}_2$	$2\mathcal{C}_3$
$\Gamma^{(1)}$	1	1	1
$\Gamma^{(2)}$	1	-1	1
$\Gamma^{(3)}$	2	0	-1

$\mathcal{C}_1 = \{E\}$
$\mathcal{C}_2 = \{A, B, C\}$
$\mathcal{C}_3 = \{D, F\}$

Note: Two funny properties of this table:

1. The columns are perpendicular to each other.

2. The rows are perpendicular if using the number of elements as a metric.

This is a general feature that can be proved using the *orthogonality theorem*.

To prove this theorem we first need two lemmas:

Lemma 1: Any matrix representation with determinant $\neq 0$ is equivalent to a *unitary* representation.

Proof. Consider any representation $A_i \in G$ (writing A_i instead of $\Gamma(A_i)$ just to speed up writing) then

$$H = \sum_{i=1}^{g} A_i A_i^\dagger = H^\dagger$$

(i.e. H is hermitian). Any hermitian matrix can be diagonalised by a unitary matrix.

$$\exists U \text{ such that } d = U^{-1} H U$$

where d is diagonal and all d_{ii} are real and *positive*. (This is an exercise.) Write d in terms of $A'_i = U^{-1} A_i U$.

$$\Rightarrow \quad \mathbf{1} = d^{-1/2} \sum_i A'_i A_i'^{\dagger} d^{-1/2}$$

$$A'_i = U^{-1} A_i U$$

and hence

$$A''_j = d^{-1/2} A'_j d^{1/2}$$

is unitary.

Check:

$$A''_j \left(A''_j\right)^{\dagger} = d^{-1/2} A'_j d^{1/2} d^{1/2} \left(A'_j\right)^{\dagger} d^{-1/2} =$$

$$= d^{-1/2} A'_j d^{1/2} \left(d^{-1/2} \sum_i A'_i A_i'^{\dagger} d^{-1/2} \right) d^{1/2} \left(A'_j\right)^{\dagger} d^{-1/2} =$$

$$= d^{-1/2} A'_j \sum_i A'_i \left(A'_i\right)^{\dagger} \left(A'_j\right)^{\dagger} d^{-1/2} = d^{-1/2} \sum_i \left(A'_j A'_i\right) \left(A'_j A'_i\right)^{\dagger} d^{-1/2} =$$

(Rearrangement theorem)

$$= d^{-1/2} \sum_k A'_k \left(A'_k\right)^{\dagger} d^{-1/2} = \mathbf{1}. \qquad \square$$

Lemma 2: Schur's lemma.

A matrix which commutes with all matrices in an *irreducible* representation is constant (proportional to the identity matrix).

Proof. From previous lemma about the unitarity, we may always consider a unitary representation. Then let M be a matrix such that $M A_i = A_i M \quad \forall A_i \in G$. Take dagger on this:

$$A_i^{\dagger} M^{\dagger} = M^{\dagger} A_i^{\dagger}$$

$A_i^{\dagger} = A_i^{-1}$. Take $A_i (...) A_i \Rightarrow$

$$M^{\dagger} A_i = A_i M^{\dagger}$$

But then the two hermitian matrices:

$$H_1 = M + M^{\dagger}, \quad H_2 = \mathrm{i}\left(M - M^{\dagger}\right)$$

also commute with $\forall A_i$. Next we show that such matrices are constant.

Since H is hermitian there is a unitary matrix U such that

$$d = U^{-1} H U$$

$$\Rightarrow \quad A'_i d = d A'_i$$

where $A'_i = U^{-1} A_i U$. (Check).

In components this reads

$$\left(A_i'\right)_{\mu\nu} d_{\nu\nu} = d_{\mu\mu} \left(A_i'\right)_{\mu\nu}$$

$$\Rightarrow \quad \left(A_i'\right)_{\mu\nu} (d_{\nu\nu} - d_{\mu\mu}) = 0 \quad \forall A_i$$

So if $d_{\nu\nu} \neq d_{\mu\mu}$ then $\left(A_i'\right)_{\mu\nu} = 0 \forall i$, i.e. A' is in block form (i.e. reducible), and if the representation is irreducible then $d_{\nu\nu} = d_{\mu\mu} \Rightarrow d \propto \mathbf{1}$. □

Example: Suppose

$$d = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

i.e. $d_{11} = d_{22} = d_{33} \neq d_{44} = d_{55} = d_{66}$

$$\Rightarrow \forall A_i \quad A_i' = \begin{pmatrix} \times & \times & \times & 0 & 0 & 0 \\ \times & \times & \times & 0 & 0 & 0 \\ \times & \times & \times & 0 & 0 & 0 \\ 0 & 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times & \times \end{pmatrix}$$

Lemma 3:

If M exists such that for two irreducible representations $\Gamma^{(i)}$ and $\Gamma^{(j)}$ of dimension l_i and l_j and

$$M\Gamma^{(i)}(A_k) = \Gamma^{(j)}(A_k) M \quad \forall A_k \in G$$

then (1) if $l_i \neq l_j$ then $M = 0$, i.e. $\Gamma^{(i)}$ and $\Gamma^{(j)}$ are different irreducible representations.

(2) If $l_i = l_j$ then $M = 0$ and $\Gamma^{(i)}$ and $\Gamma^{(j)}$ are inequivalent representations or $\det(M) \neq 0$ ($\Gamma^{(i)}$ and $\Gamma^{(2)}$ are equivalent).

Proof. Similar to the proofs above. □

Theorem: The Great Orthogonality theorem.

Consider *all inequivalent irreducible* unitary representations of G denoted $\Gamma^{(i)}(R)$, then:

$$\sum_{R \in G} \left(\Gamma^{(i)}(R)\right)^*_{\mu\nu} \left(\Gamma^{(j)}(R)\right)_{\alpha\beta} = \frac{g}{l_i} \delta_{ij} \delta_{\mu\alpha} \delta_{\nu\beta}$$

Proof.

First step: Consider two inequivalent representations $\Gamma^{(1)}$ and $\Gamma^{(2)}$. Then

$$M \equiv \sum_R \Gamma^{(2)}(R) \, X \, \Gamma^{(1)}\left(R^{-1}\right)$$

where X is an arbitrary matrix. It satisfies

$$M \, \Gamma^{(1)} = \Gamma^{(2)} M$$

Check:

$$\Gamma^{(2)}(S)\,M=\sum_R \underbrace{\Gamma^{(2)}(S)\,\Gamma^{(2)}(R)}_{=\Gamma^{(2)}(SR)}X\,\Gamma^{(1)}(R^{-1})\Gamma^{(1)}(S^{-1})\Gamma^{(1)}(S)=$$

$$=\sum_R \Gamma^{(2)}(S\,R)\,X\,\Gamma^{(1)}\Big((S\,R)^{-1}\Big)\,\Gamma^{(1)}(S)=$$

(Rearrangement theorem)

$$=\sum_R \Gamma^{(2)}(R)X\,\Gamma^{(1)}(R^{-1})\,\Gamma^{(1)}(S)=M\,\Gamma^{(1)}(S).$$

This is true for any matrix X, in the definition of M.

But since we assume that $\Gamma^{(1)}$ and $\Gamma^{(2)}$ are inequivalent $M=0$ (by the previous lemma).

$$M_{\alpha\mu}=0=\sum_R\sum_{\bar\beta,\bar\nu}\Big(\Gamma^{(2)}(R)\Big)_{\alpha\bar\beta}X_{\bar\beta\bar\nu}\Big(\Gamma^{(1)}\big(R^{-1}\big)\Big)_{\bar\nu\mu}.$$

Now pick $X_{\bar\beta\bar\nu}=0$ except for $X_{\beta\nu}=1$.

$$\Rightarrow\sum_R\Big(\Gamma^{(2)}(R)\Big)_{\alpha\beta}\Big(\Gamma^{(1)}\big(R^{-1}\big)\Big)_{\nu\mu}=0.$$

By unitarity this reads

$$\sum_R\Big(\Gamma^{(2)}(R)\Big)_{\alpha\beta}\Big(\Gamma^{(1)}(R)\Big)^*_{\mu\nu}=0.$$

This implies δ^{ij} on the right hand side of the Great Theorem.

Second step: Choose now $\Gamma^{(1)}$ and $\Gamma^{(2)}$ as equivalent representations:

$$M=\sum_R \Gamma(R)\,X\,\Gamma(R^{-1})$$

It can be checked that it commutes with all A_i in G. But then by Schur's lemma $M=c\,\mathbf{1}$ for some constant c.

$$\Rightarrow c\,\delta_{\mu\mu'}=\sum_R\sum_{\bar\nu,\bar\rho}(\Gamma(R))_{\mu,\bar\nu}X_{\bar\nu\bar\rho}\big(\Gamma\big(R^{-1}\big)\big)_{\bar\rho\mu'}$$

Then set $X=0$ except $X_{\nu\rho}=1$.

$$c_{\nu\rho}\,\delta_{\mu\mu'}=\sum_R(\Gamma(R))_{\mu\nu}\big(\Gamma\big(R^{-1}\big)\big)_{\rho\mu'}$$

The c constant depends on the choice of X. To get $c_{\nu\rho}$ we sum over $\mu=\mu'$:

$$\underbrace{c_{\nu\rho}\sum_\mu\delta_{\mu\mu}}_{l}=\sum_R\underbrace{\sum_\mu\big(\Gamma\big(R^{-1}\big)\big)_{\rho\mu}(\Gamma(R))_{\mu\nu}}_{=(\Gamma(R^{-1}R))_{\rho\nu}}=\sum_R\mathbf{1}_{\rho\nu}=g\,\mathbf{1}_{\rho\nu}$$

$$\Rightarrow c_{\nu\rho}=\frac{g}{l}\,\delta_{\nu\rho}$$

Unitarity:

So

$$\sum_R (\Gamma(R))^*_{\mu'\nu'} (\Gamma(R))_{\mu\nu} = \frac{g}{l}\,\delta_{\mu\mu'}\delta_{\nu\nu'}$$

Combining the two steps the theorem is proved. □

Implications of the great theorem.

1. Viewing the numbers $(\Gamma^{(i)}(R))_{\mu\nu}$ as components of a set of vectors in a $g = |G|$-dimensional space and i, μ, ν enumerate the different vectors.

$$\Rightarrow \sum_i (l_i)^2 \leqslant g$$

since the left hand side is $\sum_{i,\mu,\nu}$ = all g-dimensional vectors, which are all orthogonal to each other by the Great Theorem. In fact we will later show that the equality is true:

$$\sum_i (l_i)^2 = g$$

EXAMPLE: $|D_3| = 6 = 1^2 + 1^2 + 2^2$ is the unique solution as we will see below.

2. Orthogonality for characters:

a. Set $\mu = \nu$ $(\chi\Gamma^{(i)})$ and $\alpha = \beta$ $(\chi\Gamma^{(j)})$ and $\sum_{\mu,\alpha}$

$$\Rightarrow \sum_R \chi^{(i)}(R)^* \chi^{(j)}(R) = \sum_{\mu,\alpha} \frac{g}{l_i}\delta_{ij}\,\underbrace{\delta_{\mu\alpha}\delta_{\mu\alpha}}_{=\delta_{\mu\mu}\to l_i} = g\,\delta_{ij}$$

$$\Rightarrow \sum_k \chi^{(i)}(\mathcal{C}_k)^* \chi^{(j)}(\mathcal{C}_k)\, N_k = g\,\delta_{ij}$$

where N is the number of elements in $\mathcal{C}_k$.

1) i.e. the rows in the character table are orthogonal.

2) Number of irreducible representations = number of classes.

$$|D_3| = 6 = 1^2 + 1^2 + 2^2$$

b. Form the square matrix

$$Q = \begin{pmatrix} \chi^{(1)}(\mathcal{C}_1) & \chi^{(1)}(\mathcal{C}_2) & \cdots \\ \chi^{(2)}(\mathcal{C}_1) & \cdots & \\ \vdots & & \end{pmatrix}$$

and consider

$$Q' = \frac{1}{g}\begin{pmatrix} \chi^{(1)}(\mathcal{C}_1)^* N_1 & \chi^{(2)}(\mathcal{C}_1)^* N_1 & \cdots \\ \chi^{(1)}(\mathcal{C}_2)^* N_2 & & \\ \vdots & & \end{pmatrix}$$

Can check

$$(Q\,Q')_{ij} = \sum_k \frac{\chi^{(i)}(\mathcal{C}_k)\chi^{(j)}(\mathcal{C}_k)^* N_k}{g} = \delta_{ij}$$

$$\Rightarrow Q' = Q^{-1}$$

$$\Rightarrow (Q'\,Q)_{ij} = \delta_{ij}$$

$$\Rightarrow \sum_i \chi^{(i)}(\mathcal{C}_k)^* \chi^{(i)}(\mathcal{C}_l) = \frac{g}{N_k}\,\delta_{kl}$$

i.e. the columns in the character table are orthogonal.