## 2009–09–07

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Let's start by discussing finite groups.

### Chapter 1: Finite groups

Definition: A group $G$ is a set $(\underbrace{\text{finite or infinite}}_{\text{discrete cases}}, \ldots)$ of elements $g \in G$ with a *composition law* satisfying the following axioms:

i. If $g_1 \in G, g_2 \in G$, then $g_1 \cdot g_2 = g_3 \in G$. (It is a closed set, in this sense.)

ii. It is *associative*: for $g_1, g_2, g_3 \in G$, then

$$g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3$$

where brackets tell you the order of the operations.

(Example of sets not satisfying this: octonions.)

iii. There exists a *unit* $e \in G$ which is unique, such that $e\,g = g\,e = g$ for all $g \in G$.

iv. For any $g \in G$ there exists a unique element $g^{-1} \in G$ such that $g\,g^{-1} = g^{-1}\,g = e$.

Exercise 1. Show that in (iv) one can demand $g\,g^{-1} = e$ and derive $g^{-1}\,g = e$.

Exercise 2. Show that the uniqueness of $g^{-1}$ can be derived instead of postulated.

Exercise 3. Rewrite $(g_1 g_2)^{-1}$ as a product of inverses if $g_1\,g_2 \neq g_2\,g_1$.

Read FS § 4.2.

### Overview of *simple* groups

Simple groups cannot be split into a sum of sets of smaller groups. Simple groups can be completely classified (except in some complicated cases).

**A.** Finite groups.

Finite groups have a finite number of elements. The number of elements is called the *order*. (There is a wider class: discrete groups, with finite or infinite number of elements.)

Complete classification:

- 4 infinite series
- 26 sporadic cases. (Example: Monster.)

**B.** Lie groups.

The elements depend on a number of continuous parameters. The number of continuous parameters is called the dimension of the group.

**B1.** dim = finite

This include the rotation group in space, parametrised by three Euler angles.

Cartan classification:

- 4 infinite series: $A_n, B_n, C_n, D_n$. These are called classical.

- Exceptional groups: $G_2, F_2, E_6, E_7, E_8$.

**B2.** dim = infinite

Sometimes these are only known as Lie algebras.

Kac–Moody, Virasoro, (critical phenomena, phase transitions, string theory)

Diff($M$), (general relativity)

Borcherds algebras.

**Lie groups are *smooth* manifolds.**

EXAMPLE: SU(2).

A $2 \times 2$ matrix with complex entries, satisfying $U^{\dagger}U = 1 \Rightarrow \det U = \pm 1$. If $\det U = +1$ we have SU(2).

$$U = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \quad \begin{cases} \alpha = x_1 + \mathrm{i}x_2 \\ \beta = x_3 + \mathrm{i}x_4 \end{cases}$$

$$\det U = |\alpha|^2 + |\beta|^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$$

This is $S^3$.

*Note:* $S^3$ is *simply connected*.

What is the group SO(3)?

$$\underbrace{\begin{pmatrix} 3\times 3 \\ \text{real} \end{pmatrix}}_{R} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}, \quad R^T R = 1$$

$\mathbb{R}P^3$.

**Figure 1.** Half $S^3$ with cross cap.

EXERCISE 4: Make a double-loop across the cross cap trivial.

**Figure 2.** Trivial and non-trivial loops on half $S^3$ with cross cap.

(Read FS § 9.1)

*Comment:* In quantum mechanics we deal with representations of SU(2) (i.e. its Lie algebra) of spin $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$

But for the groups:

SU(2): all $j$ values possible (tensors and spinors)
SO(3): only $j$ integer possible (tensors)

**Finite groups** (see Tinkham)

We will now introduce a number of concepts and derive some theorems. Consider the following example of an *abstract finite group* called $D_3$.

$$D_3 = \{E, A, B, C, D, F\}$$

($E$ is the unit element) with multiplication table

| $i\backslash j$ | $E$ | $A$ | $B$ | $C$ | $D$ | $F$ |
|---|---|---|---|---|---|---|
| $E$ | $E$ | $A$ | $B$ | $C$ | $D$ | $F$ |
| $A$ | $A$ | $E$ | $D$ | $F$ | $B$ | $C$ |
| $B$ | $B$ | $F$ | $E$ | $D$ | $C$ | $A$ |
| $C$ | $C$ | $D$ | $F$ | $E$ | $A$ | $B$ |
| $D$ | $D$ | $C$ | $A$ | $B$ | $F$ | $E$ |
| $F$ | $F$ | $B$ | $C$ | $A$ | $E$ | $D$ |

Entry $g_{ij} \equiv g_i \cdot g_j$.

EXAMPLE: $A\,B = D \neq B\,A = F$. So $D_3$ is *non-abelian*.

DEFINITION. Abelian iff $g_1\,g_2 = g_2\,g_1\ \forall\, g_1, g_2 \in G$.

EXAMPLE: $g_1(g_2\,g_3) = (g_1 g_2) g_3 \forall g_i \in G$. Can be checked.

DEFINITION:The number of elements $= |G| =$ order, is called the *order*. ($|D_3| = 6$).

*Comment:* The whole group $D_3$ can be *generated* by two nontrivial ($\neq E$) elements.

This abstract group can be realised in many ways. Here are two:

1. As symmetry operations acting on an equilateral triangle:

   **Figure 3.** Fixed numbers on the corners. $A, B, C$ are space-fixed axes.

   $E$: trivial (no operation).

   $A, B, C$: flips around the corresponding axis.

   $D, F$: $2\pi/3$ and $-2\pi/3$ rotation $\perp$ the triangle.

   $A\,B = D$.

   $B$: ${}_2\Delta_1^3 \rightarrow {}_2\Delta_3^1$

   $A$: ${}_2\Delta_3^1 \rightarrow {}_3\Delta_2^1$

   $D$: ${}_2\Delta_1^3 \rightarrow {}_3\Delta_2^1$

2. Matrices.

   Consider the matrices:

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$B = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \quad C = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

$$D = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \quad F = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

   Again $A\,B = D$ etc.

   These matrices are called a (matrix) *representation* of $D_3$.

In this case of $2 \times 2$ matrices all elements in $D_3$ are different $\Rightarrow$ faithful representation. Note that $E = D = F = 1$, $A = B = C = -1$ ($1 \times 1$ matrices) satisfy the multiplication table. In this case the map from $D_3$ to $\{1, -1\}$ is an *homomorphism*. When the representation is faithful the map is an *isomorphism.*

Homomorphism: many $\rightarrow$ one map.

Isomorphism: one $\rightarrow$ one map. (FS pages 65 and 54)

DEFINITION: A *representation* (here matrix) must satisfy that

$$A\,B = D \quad \underset{\Gamma}{\Longrightarrow} \quad \Gamma(A) \cdot \Gamma(B) = \Gamma(D).$$

Some general facts:

- Rearrangement theorem.

  From the multiplication table of $D_3$ we see that each row and each column contains every element once $\rightarrow$ i.e. just a rearrangement.

**Proof.** Fix $A_k$, then the set of elements $E\,A_k$, $A_1\,A_k$, $A_2 A_k$, ..., $A_{|G|} A_k$ will contain any element in $G$ since for any $A_i$ to appear in this list we use an $A_r$ such that $A_r A_k = A_i$ which always exists since $A_r = A_i A_k^{-1}$ and since the number of elements is the same the theorem follows. □

- If a subset of elements $H$ of those in $G$ satisfy the group axioms it is called a subgroup of $G$.

EXAMPLE: For any element in $D_3$ we can form cyclic subgroup

$$\{E, X, X^2, X^3, ..., X^{n-1}\}$$

such that $X^n = E$.

In $D_3$, if $X = A$ then $n = 2$. $A^2 = E$.

If $X = D$ then $n = 3$. $D^3 = E$.

**Cosets**

Let $H = \{E, B_2, ..., B_{|H|}\}$ be a subgroup of $G = \{E, A_2, ..., A_{|G|}\}$. Then $H\,x$, for any $x \in G$, refers to a set $\{E\,x,\, B_2\,x,\, B_3\,x,\, ...,\, B_{|H|}\,x\}$ called the *right coset*. The *left coset* is denoted $x\,H$ and works in the same way. Note: if $x \in H$ the cosets are just $H$ themselves. (Rearrangement theorem.) If $x \notin H$ then $x\,H$ and $H\,x$ are not groups, since the unit element $E$ does not appear in these cosets.

In fact: $H$ and $H\,x$ for $x \notin H$ are disjoint sets.

**Proof.** Assume the opposite:

$$\underbrace{B_i}_{\in H} \cdot \underbrace{x}_{\notin H} = \underbrace{B_j}_{\in H}$$

$\Rightarrow x = B_i^{-1} B_j \in H$. Contradiction. □

Notation: $G/H$ is the set of left cosets. (Lie algebra: $G/H$ is a coset space.)

Hence: $G$ can be divided into a set of distinct subsets.

$$G = \{H, H\,x_2, H\,x_3, ..., h\,x_l\}$$

where $l$ is the number of (distinct) cosets.

$$|G| = |H| \cdot l$$

EXAMPLE: $|D| = 6 = 3 \cdot 2$.

— $|H| = 3, l = 2$. (This is the rotation.)

— $|H| = 2, l = 3$. (This is the flip.)