Examination Problems

Gravitation and Cosmology

Tensors and Geodesics

1. Describe in one sentence what the orbit of the Moon, the trajectory of a falling apple and the trajectory of a bullet have in common according to Einstein.

2. Assume that $A^\mu B_{\mu\nu}$ is a covariant vector for all contravariant vectors A^μ. Show that $B_{\mu\nu}$ transforms as a tensor. Also prove that $g^{\mu\nu}$, defined as the inverse of the metric $g_{\mu\nu}$ is a second rank contravariant tensor. (Hint: Assume that it does transform as a contravariant tensor, and use the fact that the inverse of a matrix is unique.)

3. A fluid is consisting of point particles of rest mass m moving to the right along the x-axis with a gamma-factor of 2 and number density n, as seen from a certain coordinate system. What is the (space-averaged) energy momentum tensor for this configuration? If one also introduces a second set of particles , identical to the ones described above, but moving moving to the left *through* the right moving particles, what then is the energy momentum tensor?

4. Consider the Levi-Civita symbol, or the epsilon symbol, defined by $\epsilon^{0123} = 1$ and total antisymmetry in the four indices. Show that $\epsilon^{\mu\nu\kappa\lambda}$ is a tensor density of weight -1.

   Show that the invariant integration measure is $d^4x\sqrt{-\det g}$, i.e., that

   $$\int d^4x\sqrt{-\det g}\,\phi(x)$$

   is independent of the choice of coordinates when ϕ is a scalar.

5. Determine all time-like geodesics and the affine connection for the space-time described by the two-dimensional metric

   $$d\tau^2 = \frac{1}{t^2}dt^2 - \frac{1}{t^2}dx^2.$$

6. Determine all geodesics for the two-dimensional metric,

   $$d\tau^2 = t^4 dt^2 - t^2 dx^2\,.$$

7. Three-dimensional anti-deSitter space (AdS_3) is described by the metric

   $$ds^2 = -du^2 - dv^2 + dx^2 + dy^2,$$

   where the coordinates are confined to the hyperboloid

   $$-u^2 - v^2 + x^2 + y^2 = -b^2.$$

Change coordinates according to

$$
\begin{aligned}
u &= \sqrt{b^2+r^2}\cos\frac{t}{b} \\
v &= \sqrt{b^2+r^2}\sin\frac{t}{b} \\
x &= r\cos\phi \\
y &= r\sin\phi
\end{aligned}
$$

(check that this defines a valid set of coordinates on the hyperboloid). Using the metric in the new coordinates, calculate the proper distance from the origin $r = 0$ to spatial infinity $r \to \infty$. Also find the coordinate time needed for a photon to travel this distance.

Curvature

8. Does the metric

$$d\tau^2 = dt^2 - dx^2 - dy^2 - dz^2 + 4\cosh\frac{x}{2}[\cosh\frac{x}{2}(dt+dx) - \sinh\frac{x}{2}dy]dx$$

correspond to a curved space-time? How many positive and negative eigenvalues does it have?

9. Does the metric

$$d\tau^2 = (1-x^2)dt^2 - dx^2 - dy^2 - x^2dz^2 - 2x^2dtdz$$

correspond to a curved space-time?

10. For a certain space-time of arbitrary dimension D, the curvature tensor has the form

$$R_{\mu\nu\rho\sigma} = f(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}),$$

where f is a scalar function. Show that this curvature tensor has the appropriate symmetries. Show that f must be a constant if $D \geq 3$. For which value of the cosmological constant Λ is such a space-time a solution to Einstein's equations?

11. Calculate the Ricci tensor and the curvature scalar for the metric

$$d\tau^2 = e^{2\alpha}dt^2 - e^{2\beta}dx^2$$

where α and β are arbitrary functions of t and x.

12. As an example of a space with constant negative curvature, take the simplest example, the two-dimensional "Poincaré disk". Its metric can be written as

$$ds^2 = \frac{dr^2 + r^2d\phi^2}{(1-\frac{r^2}{a^2})^2},$$

where $0 \leq r < a$ and $0 \leq \phi < 2\pi$ (ϕ is an angle). What is the distance from a point with coordinates (r, ϕ) to the "boundary" $r = a$? Calculate the curvature tensor (note that in two dimensions, there is only one independent component, given the properties on p. 141, so it is enough to calculate R_{1212}), the Ricci tensor and the curvature scalar. Compare to the corresponding results for a two-dimensional sphere.

13. Consider the d-dimensional geometries described by the metrics

$$
\begin{aligned}
&(1) \qquad ds^2 = \tfrac{-dt^2 + \delta_{ij} dx^i dx^j}{t^2}\,, \\
&(2) \qquad ds^2 = \tfrac{dy^2 + \eta_{\alpha\beta} dx^\alpha dx^\beta}{y^2}\,,
\end{aligned}
$$

where δ and η are metric tensors for flat euclidean and minkowskian $(d-1)$-dimensional space, respectively. Calculate the curvature tensor in both cases, and describe the spaces in words.

The Schwarzschild Solution

14. Is it possible to see oneself from behind, standing outside a black hole? In other words, are there any closed photon orbits?

15. Two identical atoms are at rest at radii R_1 and R_2 in a gravitational field with Schwarzschild metric. When observed locally, these atoms emit radiation at frequency ν. An observer at a very large distance ($r \to \infty$) measures frequencies ν_1 and ν_2, respectively. Find the ratio ν_1/ν_2.

16. An astronaut is orbiting the earth in a spaceship. She has chosen the altitude of her (circular) orbit carefully, in order to make her on-board clock tick at the same rate as a clock at rest on the surface of the earth. Neglecting the rotation of the earth, calculate this altitude.

17. Determine the metric and the electric field outside a *charged* black hole, i.e., find a static, spherically symmetric solution to Maxwell's equations and Einstein's equations with the electromagnetic energy-momentum tensor. Under what circumstances does this metric (the Reissner-Nordström metric) have a Schwarzschild singularity? Does a proton have such a singularity?

18. (A variation on the previous problem.) Determine the metric and the electric field outside a charged black hole, *i.e.*, find a static, spherically symmetric solution to Maxwell's equations and Einstein's equations with the electromagnetic energy-momentum tensor. (The solution should contain two parameters, the mass M and the electric charge Q. They may be checked *e.g.* by inspecting the asymptotic behaviour of the solution at spatial infinity.) Show that, unless $M > Q$ (in suitable units), the solution has a singularity that is not surrounded by a horizon (similar to a Schwarzschild solution with negative mass). Such solutions are expected to be "forbidden"; this is the so called "cosmic censorship principle".

19. A rotating black hole is described in the plane $\theta = \frac{\pi}{2}$ by the metric

$$d\tau^2 = Adt^2 + 2Cdtd\phi - Dd\phi^2 - Bdr^2,$$

where $A = (1 - \frac{a}{r})$, $C = \frac{ab}{r}$, $D = r^2 + b^2(1 + \frac{a}{r})$ and $B = \frac{r^2}{r^2 - ar + b^2}$. In the case of the Schwarzschild solution $a = 2MG$, while b is related to the total angular momentum of the black hole. Consider a particle, initially at rest far away from the black hole, that starts to fall towards it. Find an expression for $\frac{dr}{d\tau}$ in terms of r. Show also that when the particle approaches the black hole it will obtain an angular velocity $\frac{d\phi}{d\tau}$ which is different from zero.

20. Derive the Schwarzschild metric in a space-time with *arbitrary* dimensionality $D \geq 4$. Using this result, investigate whether stable circular orbits exist in this geometry.

21. An astronaut has just landed on a small spherical asteroid of radius $R = 5$ km. The astronaut height is $d = 1.70$ m and the height of the spaceship is $D = 15$ m. The astronaut can walk away from the spaceship a distance L (measured on the ground) before she loses sight of the spaceship simply due to the curvature of the planet. Compute first L by elementary geometry ignoring any gravitational effect. Now imagine the rather absurd situation where the planet is replaced by a neutron star of mass $M = 2 \cdot 10^{30}$ kg of the same radius R. Assume that the astronaut and the spaceship will not be crushed by the gravity of the star. Including the effect of bending of the light compute the new distance L' that she can travel walking away from the spaceship before losing sight of it.

22. On a neutron star of radius $R = 5$ km and mass $M = 2 \cdot 10^{30}$ kg two photons both of wavelength $\lambda = 300$ nm are emitted 10 meters from the surface. One photon heads towards the surface of the star and the other escapes towards the Earth. Compute the wavelength of the photon reaching the surface as measured by an observer on the surface of the star and the wavelength of the escaping photon as measured by an observer on Earth.

23. A beacon radiating at a fixed frequency ν_0 is released at time $t = 0$ towards a black hole of mass M by an observer situated very far away from the black hole. The observer stays at a constant distance from the black hole while the probe is falling. Show that the frequency of the beacon (when it is close to the event horizon) as measured by the observer can be written as $\nu \sim e^{-\frac{t}{K}}$ for some constant K and relate the constant K to the mass of the black hole.

24. A rocket is orbiting on a perfectly circular orbit around a black hole. To remain on a fixed radius it needs a radial rocket-thrust, and thus a proper acceleration. How does the proper acceleration α depend on the radius r and the angular velocity Ω of the rocket? ($\Omega = d\varphi/dt$ in the usual coordinates.) Verify that at $r = 3MG$, the acceleration is independent of Ω. (Hint: $\alpha^2 = g_{\mu\nu}\frac{Du^\mu}{D\tau}\frac{Du^\nu}{D\tau}$, where D means covariant differentiation along a curve.)

25. A clock is shot up vertically from the surface of the Earth until it reaches a maximal height $h = 30.000$ km and then falls down again. Ignoring the rotation of the Earth, compute the total time delay between this clock and an identical clock on Earth at the end of the journey.

26. Determine the deflection of a *massive* particle entering the solar system with velocity v, gracing the sun at a distance d and thereafter leaving. v may be assumed to be large, but less than c.

27. The metric of a rotating charged black hole can be written as:

$$d\tau^2 = (1-f)dt^2 - 2af\sin^2\theta dt d\phi - \Sigma(\frac{dr^2}{\Delta} + d\theta^2) - g\sin^2\theta d\phi^2,$$

where

$$\begin{aligned} \Delta &= r^2 - (2mr - Q^2) + a^2 \\ \Sigma &= r^2 + a^2\cos^2\theta \\ f &= (2mr - Q^2)/\Sigma \\ g &= r^2 + a^2 + fa^2\sin^2\theta \\ m^2 &> Q^2 + a^2 \end{aligned}$$

and m, Q, a, are mass, charge and angular momentum respectively and we set $G_N = 1$.

For a circular orbit in the equatorial plane: $\theta = \pi/2$ derive the law corresponding to Kepler's law $\Omega^2 = m/r^3$ for an electrically neutral object with angular velocity Ω at a radius r. Compute the time dilation factor for an observer moving in such an orbit, as judged by a distant stationary observer.

28. a. The German astronaut Thomas Reiter has spent almost 6 months aboard the ISS. Calculate the accumulated difference in elapsed time on the ISS and on earth after this period of time.
b. Are general relativistic effects significant for time-keeping in the GPS system?

Symmetic Spaces and Cosmology

29. Prove that for a given space-time with a Killing vector $\xi^\mu(x)$, the scalar $\xi^\mu P_\mu$ is a constant of the motion for a freely falling particle.

30. A plane-fronted gravitational wave in the u-direction (which is light-like) may be described by a metric $ds^2 = -2dudv + a^2(u)dx^2 + b^2(u)dy^2$, where a and b are some arbitrary functions. Determine all Killing vectors from the Killing equations. Check that the infinitesimal transformations of the coordinates indeed leave the metric form-invariant. Interpret, if possible, some of the Killing vectors.

31. Consider a toroidal surface embedded in flat three-dimensional euclidean space. With (ρ,ϕ,z) being standard cylindrical coordinates and a/b being the radius of the torus/tube, it can be parametrised as $(\rho,\phi,z) = (a + b\cos\theta, \phi, b\sin\theta)$. Calculate the curvature scalar. Comment on the sign of R for different points on the torus. Finally, determine all Killing vectors in this space.

32. A four-dimensional space-time is descibed by the metric

$$ds^2 = -2dudv + dx^2 + dy^2 - \omega^2(x^2 + y^2)du^2 \,,$$

where ω is a constant. Verify that the signature is $(-1, 1, 1, 1)$. Find all Killing vectors (some, but not necessarily all, are obvious in the sense that they don't require calculation). Calculate the energy-momentum tensor for the matter or radiation that acts as a source allowing this solution.

33. A generic Friedman-Robertson-Walker cosmology is described by a metric

$$d\tau^2 = dt^2 - R^2(t)(dx^2 + dy^2 + dz^2)$$

with a generic function $R(t)$. Making a suitable coordinate transformation show that the metric can be re-written as

$$d\tau^2 = f^2(t)(dt^2 - dx^2 - dy^2 - dz^2)$$

and find the expression of $f(t)$ in terms of $R(t)$. Compute the geodesic equations in the new metric.

34. Show, or argue in a convincing way, that the three spatially homogeneous and isotropic solutions to Einstein's equations with a cosmological constant, i.e., the cases with $k = 0, \pm 1$, describe the same space-time.

35. The metric outside a straight, infinitely long cosmic string along the z-axis is

$$d\tau^2 = dt^2 - dr^2 - (1 - 8mG)r^2 d\alpha^2 - dz^2,$$

in cylindrical coordinates ($0 \leq \alpha \leq 2\pi$). Here m denotes the mass per unit length of the string. Show that the metric is flat and that a distant object, situated behind the string, yields a double image.

36. A galaxy acts a gravitational lens for a very distant quasar. The galaxy is 100 Mpcs away, and the distance to the quasar can be assumed to be much greater. The image is a ring with radius 1.0 arcsec. What is the mass of the galaxy? Express in solar masses.

37. The temperature of the background radiation of the universe is ≈ 2.7 K. The Hubble parameter is measured to be ≈ 70 (km/s)/Mpc. Suppose (somewhat contrary to the present observational status) that the universe is matter-dominated with exactly critical energy density, and has been so since atoms formed, matter became uncharged, electromagnetic radiation decoupled from matter and the universe became transparent. This transition can be assumed to have taken place at a temperature which is typical for atomic binding energies, $T \approx 5$ eV. Assume also that the universe before this decoupling was radiation-dominated. What is the age of the universe based on these measurements and assumptions, and what was its age at the time of decoupling?

38. With the assumptions made in the previous problem, show that the microwave background radiation we receive from different directions come from regions that at the time of decoupling were causally disconnected, *i.e.*, no information with a common source may have reached them during the time of existence of the universe.

(Still, the large-scale structure of the radiation is isotropic. This is the so-called "homogeneity problem" or "horizon problem": how can these regions then "know" that they should contain matter and radiation with the same density and temperature? It may be solved by "inflation": if the universe at some time underwent a very fast expansion, our visible part of the universe may come from a causally connected region. Incidentally, inflation can also solve the "flatness problem" and explain why the universe seems to be so close to being flat.)