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We are out of problems, officially, since we have been working fast. We will be doing cosmology exercises from the last exam.

37.

> “The temperature of the background radiation of the universe is ≈ 2.7 K. The Hubble parameter is measured to be $\approx 70\,(\text{km/s})/\text{Mpc}$. Suppose (somewhat contrary to the present observational status) that the universe is matter-dominated with exactly critical energy density, and has been so since atoms formed, matter became uncharged, electromagnetic radiation decoupled from matter and the universe became transparent. The transition can be assumed to have taken place at a temperature which is typical for atom binding energies, $T \approx 5\,\text{eV}$. Assume also that the universe before this decoupling was radiation-dominated. What is the age of the universe based on these measurements and assumptions, and what was its age at the time of decoupling?”

Introducing some symbols for quantities mentioned above: The cosmic microwave background radiation (CMB) temperature is $T_0 = 2.7$ K, and the Hubble parameter is $H_0 = 70$ km/s Mpc. (The index 0 means “today”). Assume that the universe has exactly critical energy density (which means that it is flat), and that the universe is matter-dominated and has been so since the time of decoupling (at $E_{\text{D}} = 5\,\text{eV}$), and that it was radiation-dominated before decoupling.

Determine the age of the universe (t_0) and the time of decoupling (t_{D})!

The temperature at decoupling was $T_{\text{D}} = E_{\text{D}}/k_{\text{B}} = 5.8 \times 10^4\,\text{K}$. The equations governing the evolution of the universe:

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}, \quad (\text{the Friedmann equation}), \quad H \equiv \frac{\dot{a}}{a}$$

$$\dot{\rho} + 3\,H(\rho + p) = 0, \quad (\text{energy conservation})$$

Equation of state: $p = w\,\rho$, where $w = 0$ for matter and $w = \frac{1}{3}$ for radiation. Using $p = \omega\rho$ and flatness ($k = 0$), we get

$$\begin{cases} H^2 = \frac{8\pi G}{3}\rho(t) \\ \dot{\rho}(t) + 3\,H\,(1+w)\,r(t) = 0 \end{cases} \tag{1}$$

$$(1\text{b}) \;\Rightarrow\; \frac{\dot{\rho}}{\rho} = -\,3\,(1+w)\,\frac{\dot{a}}{a} \quad\Rightarrow\quad \frac{1}{\rho}\,\mathrm{d}\rho = -\,3\,(1+w)\,\frac{\mathrm{d}a}{a}$$

$$\Rightarrow\quad \ln\rho = -\,3\,(1+w)\ln a + \ln C_0 \quad\Rightarrow\quad \rho(t) = C_0\,[a(t)]^{-3(1+w)}$$

$$(1a) \quad\Rightarrow\quad \frac{\dot{a}}{a} = C_1\,a(t)^{-3(1+w)/2} \quad\Rightarrow\quad \frac{\mathrm{d}a}{\mathrm{d}t} = C_1\,a(t)^{-3(1+w)/2+1}$$

$$\int \mathrm{d}a\,a^{3(1+w)/2-1} = C_1 \int \mathrm{d}t \quad\Rightarrow\quad a(t)^{3(1+2)/2} = C_1\,t + C_2$$

Setting the Big Bang at $t = 0$, i.e. $a(0) = 0$, we get $C_2 = 0$.

$$\Rightarrow\quad a(t) = C_3\,t^{2/3(1+w)} \tag{2}$$

For $0 \leqslant t \leqslant t_{\text{D}}$ we have $w = \frac{1}{3}$.

$$(2) \quad\Rightarrow\quad \dot{a}(t) = \frac{2}{3(1+w)}\cdot\frac{1}{t}\cdot a(t) = \frac{1}{2\,t}\,a(t) \quad\Rightarrow\quad H(t) = \frac{\dot{a}(t)}{a(t)} = \frac{1}{2\,t} \quad\Rightarrow\quad t_{\text{D}} = \frac{1}{2\,H_{\text{D}}}$$

where $H_{\mathrm{D}} = H(t_{\mathrm{D}})$. We cannot actually measure H_{D}. We can measure H_0. So we have to consider the next era as well, the era of matter domination ($w = 0$), before we can determine H_{D}.

$$\left(\frac{\dot{a}}{a_0}\right)^2 = H_0^2\Big[1 - 2\,q_0 + 2_0\Big(\frac{a_0}{a}\Big)\Big] \tag{3}$$

(This is Weinberg equation 15.3.3.) For $k = 0$ we have $q_0 = \frac{1}{2}$. Equation (3) is valid for all t in the interval $t_{\mathrm{D}} \leqslant t \leqslant t_0$.

$$\Rightarrow \quad \dot{a}^2 = H_0^2 \cdot \frac{a_0^3}{a} \quad \Rightarrow \quad \left(\frac{\dot{a}}{a}\right)^2 = H^2 = H_0^2\left(\frac{a_0}{a}\right)^3 \quad \forall t \geqslant t_{\mathrm{D}}.$$

Consider the CMB temperature scaling: $T \propto \frac{1}{a(t)}$.

$$\Rightarrow \quad \frac{T(t_1)}{T(t_2)} = \frac{a(t_2)}{a(t_1)}$$

In particular

$$\frac{T_{\mathrm{D}}}{T_0} = \frac{a_0}{a_{\mathrm{D}}}$$

$$\Rightarrow \quad H_{\mathrm{D}} = H_0\left(\frac{a_0}{a_{\mathrm{D}}}\right)^{3/2} = H_0\left(\frac{T_{\mathrm{D}}}{T_0}\right)^{3/2} = 7.1 \times 10^{-12}\,\mathrm{s}^{-1}$$

$$\Rightarrow \quad t_{\mathrm{D}} = \frac{1}{2\,H_{\mathrm{D}}} = 7.0 \times 10^{10}\,\mathrm{s} = 2200 \text{ years}$$

To determine t_0 we use (3) again:

$$\dot{a}^2 = H_0^2 \cdot \frac{a_0^3}{a} \quad \Rightarrow \quad \frac{\dot{a}}{a} = H_0\sqrt{\frac{a_0}{a}}$$

Substitute $x = a/a_0, \mathrm{d}x = \frac{1}{a}\,\mathrm{d}a$.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = H_0 \cdot \frac{1}{\sqrt{x}}$$

Separable:

$$\mathrm{d}x\,\sqrt{x} = H_0\,\mathrm{d}t$$

Integrate from t_{D} to t_0:

$$\int_{a_{\mathrm{D}}/a_0}^{1} \mathrm{d}x\,\sqrt{x} = \int_{t_{\mathrm{D}}}^{t_0} H_0 \mathrm{d}t \quad \Rightarrow \left[\frac{2}{3}x^{3/2}\right]_{a_{\mathrm{D}}/a_0}^{1} = H_0(t_0 - t_{\mathrm{D}})$$

$$\Rightarrow (t_0 - t_{\mathrm{D}}) = \frac{2}{3H_0}\left(1 - \left(\frac{a_{\mathrm{D}}}{a_0}\right)^{3/2}\right) = \frac{3}{3H_0}\left(1 - \left(\frac{T_0}{T_{\mathrm{D}}}\right)^{3/2}\right) = 2.9 \times 10^{17}\,\mathrm{s} = 9.3 \times 10^9 \text{ years}$$

$$\Rightarrow t_0 = 9.3 \times 10^9 \,\text{years}.$$

38.

> "With the assumptions made in the previous problem, show that the microwave background radiation we receive from different directions come from regions that at the time of decoupling were causally disconnected, *i.e.*, no information with a common source may have reached them during the time of existence of the universe.

(Still, the large-scale structure of the radiation is isotropic. This is the so-called 'homogeneity problem' or 'horizon problem': how can these regions then 'know' that they should contain matter and radiation with the same density and temperature? It may be solved by 'inflation': if the universe at some time underwent a very fast expansion, our visible part of the universe may come from a causally connected region. Incidentally, inflation can also solve the 'flatness problem' and explain why the universe seems so close to being flat.)"

In short: With the assumptions in 37, show that the CMB radiation we receive from different directions come from regions that were causally *disconnected* at the time of decoupling!

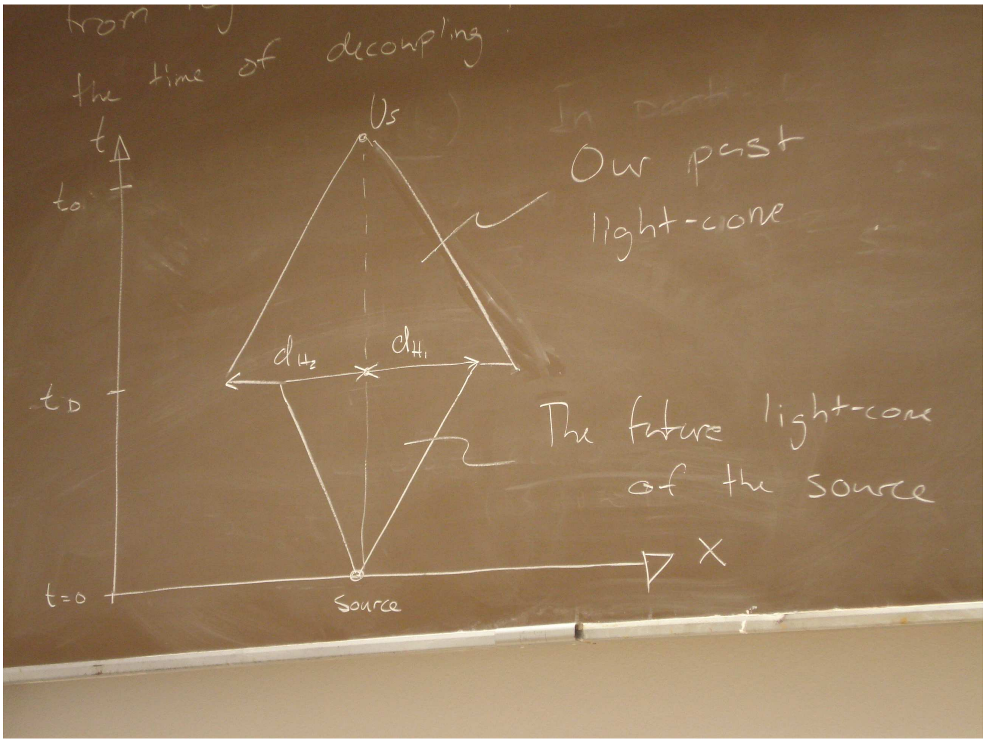

Figure 1. Our past light-cone, and the future light-cone of a source.

Consider a signal with $\dot{\theta} = \dot{\varphi} = 0$ in conformal time. $\mathrm{d}s^2 = a(\tau)\left(-\mathrm{d}\tau^2 + \mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2\right)$. $r_H = \tau_2 - \tau_1$. Convert to proper distance:

$$d_H = a(\tau_2)\,(\tau_2 - \tau_1) = a(t_2)\int_{t_1}^{t_2} \frac{\mathrm{d}t'}{a(t')}$$

From the previous problem

$$a(t) = a_\mathrm{D}\left(\frac{t}{t_\mathrm{D}}\right)^{2/3(1+w)}, \quad w = \frac{1}{3} \text{ for } 0 \leqslant t \leqslant t_\mathrm{D}.$$

$$a(t) = a_0\left(\frac{t}{t_0}\right)^{2/3(1+w)}, \quad w = 0 \text{ for } t_\mathrm{D} \leqslant t \leqslant t_0.$$

$$d_{H_1} = a(t_\mathrm{D})\int_0^{t_\mathrm{D}} \frac{1}{a_\mathrm{D}}\left(\frac{t_\mathrm{D}}{t'}\right)^{1/2} \mathrm{d}t' = t_\mathrm{D}^{1/2}\left[2\,t^{1/2}\right]_0^{t_\mathrm{D}} = 2\,t_\mathrm{D}$$

$$d_{H_2} = a(t_0)\int_{t_\mathrm{D}}^{t_0} \frac{1}{a_0}\left(\frac{t_0}{t'}\right)^{2/3} \mathrm{d}t' = t_0^{2/3}\left[3\,t^{1/3}\right]_{t_\mathrm{D}}^{t_0} = 3t_0 - 3\,t_0^{2/3}\,t_\mathrm{D}^{1/3}$$

The condition for the regions to be causally disconnected is simply $d_{H_2} > d_{H_1}$. With $t_0 = 9.3 \times 10^9$ years and $t_\mathrm{D} = 2.2 \times 10^3$ years, this condition is satisfied. The regions are causally disconnected!