2008–12–09

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Yesterday we looked at solutions to Einstein's equations describing the evolution of the universe. I remind you again of the assumptions we put in. The basic assumption was that the universe is homogeneous and isotropic. Or rather, there exists a choice of time, of time slices, where the space part has these properties. This is a strong assumption, but it enabled us to write down a solution where the universe was either spherical, flat or hyperbolic.

This is true to a very high degree, but as we mentioned yesterday, we know that we can't see the whole universe. That's a fact. If we draw a picture with conformal time and radius, say, we see light rays go like this ⟨fig⟩. Something happening outside our light cone is unknown to us. The real, large scale universe, is not available to us. Thus, the assumptions we put into our ansatz consist of an enormous extrapolation of what we can know. Nevertheless, the fact that the universe was once very dense is a safe one.

$k=1$: closed (spherical). That results in a recollapse, a big crunch.

$k=0,-1$: flat and hyperbolic. These give rise to eternal expansion.

But as we will see, as soon as we introduce the cosmological constant, this conclusion is not true anymore. The reason we talk about this here, is that this seems to be a part of the real universe. So, what is a cosmological constant?

Cosmological constant

It is a possible term in Einstein's equations. When we derived these we had a few assumptions: general covariance (tensors on both sides). We had $R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=-8\pi G\,T_{\mu\nu}$. This was essentially unique. It is unique when we say something more than general covariance: We wanted to have second derivatives. That is nice, because it provides us with the Newtonian limit. But we could relax the requirement of second order. We could have higher order derivatives — as long as we write down tensors. Such modifications are often discussed in connection to string theory. String theory gives rise to several subtle corrections. But we could also go down, to fewer derivatives. $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \Lambda g_{\mu\nu} = -8\pi G\, T_{\mu\nu}$. The sign before Λ is only convention [and Martin Cederwall is unsure of which convention he wants to use]. Λ is called the cosmological constant. Einstein introduced this constant himself when he developed his theory. He realised that when he tried to look at the large-scale stuff that his equations did not allow for a static universe, which was catastrophic. No one had ever considered the possibility of a non-static universe. A hundred years ago, the expansion was not known at all. Einstein tried to modify his equations to add a force which would prevent the universe from evolving. Then he took it away again, and thought this was an embarrassing mistake. It is unclear why he thought this was such a mistake, I don't feel he should have been embarrassed.

We could see what happens when we try to solve Einstein's equations with this term. We could also try to move it to the other side of the equations, and think of it as a modified energy-momentum tensor, like this:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} = -8\pi G\left(T_{\mu\nu} - \frac{\Lambda}{8\pi G}g_{\mu\nu}\right)$$

[Martin chases minus signs and conventions. He calls Weinberg's sign convention an abnormal convention, and warns that the material about cosmology linked from the course home page has a different sign convention for the Riemann curvature tensor.]

This means that we have an energy density ρ and a pressure p:

$$\rho = \frac{\Lambda}{8\pi G}, \quad p = -\frac{\Lambda}{8\pi G}$$

Think of Λ as positive. Then we have a positive energy density, and *negative* pressure. You might draw the conclusion — which is wrong — that a negative pressure would tend to pull the universe back together, but it is the other way around. We will discuss this when solving the equations. Anyway, the parameter $w = -1$ here. (Remember $w = 0$ for dust, and $w = \frac{1}{3}$ for radiation.) This behaves very differently. It is like an energy that resides in the vacuum itself.

Let's look at the equations. We solve the equations for the case where $T_{\mu\nu} = 0$, so that we do not mix different values of w. That is not something to be done analytically (you can do it numerically). Normally, some type of energy dominates, and since Λ is constant, this one does not get diluted as the universe expands. We just get more of it, in total. The density remains the same.

We use the equations we had before, noting that at least some of them do not hold for this particular value of w.

The Friedmann equation:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \cdot \frac{\Lambda}{8\pi G} - \frac{k}{a^2}$$

We are keeping all the three possibilities for k: flat ($k = 0$), spherical ($k = 1$) and hyperbolic ($k =$

-1). This is not a difficult equation to solve:

$$\dot{a}^2 = -k + \frac{\Lambda}{3} a^2$$

Take the case when $\Lambda > 0$. If $\Lambda > 0$ we have all three possibilities for k. If Λ is negative, since the left hand side is positive, we can only have positive k. (Remember how we reasoned about the critical density ρ_c.) Positive cosmological constant means

$$k = 1 \quad \Rightarrow \quad a(t) = \sqrt{\frac{3}{\Lambda}} \cosh\left(\sqrt{\frac{\Lambda}{3}}\, t\right)$$

There should be some integration constant also, but we can shift time by some constant. We don't write any constant and let the time $t = 0$ be determined by the above. As $t \to \infty$ this expands exponentially.

$$k = 0 \quad \Rightarrow \quad a(t) = \sqrt{\frac{3}{\Lambda}} \exp\left(\sqrt{\frac{\Lambda}{3}}\, t\right)$$

You might have expected something different, but this also goes exponentially for large t.

$$k = -1 \quad \Rightarrow \quad a(t) = \sqrt{\frac{3}{\Lambda}} \sinh\left(\sqrt{\frac{\Lambda}{3}}\, t\right)$$

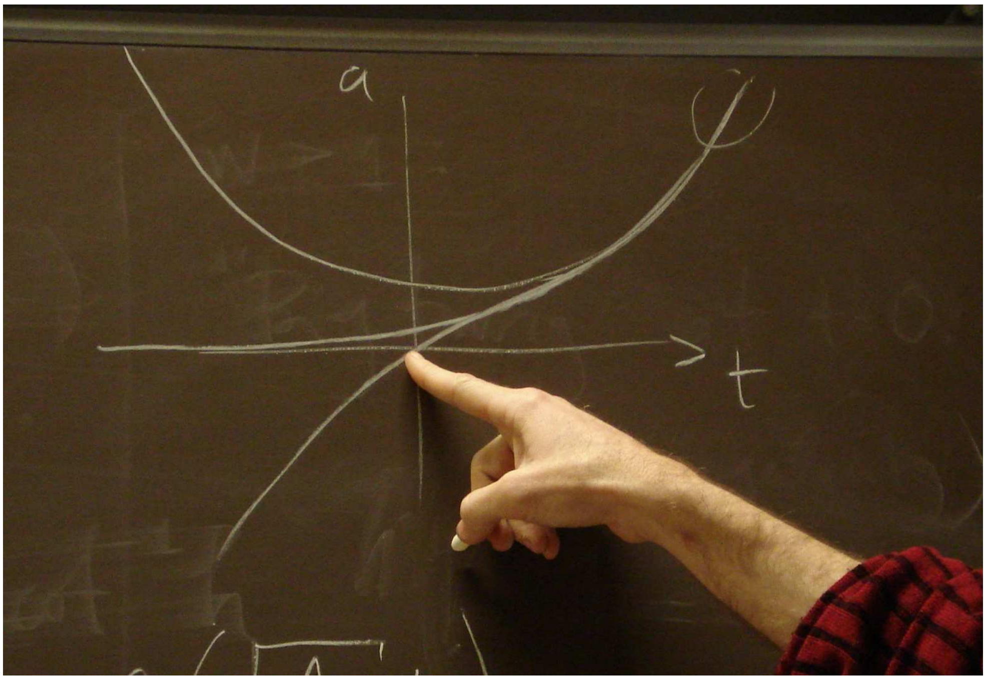

Figure 1. The solutions for different values of k look similar for large t, but closer to $t = 0$ they look very different. The $k = -1$ solution, $a(t) = \sqrt{\frac{3}{\Lambda}} \sinh\left(\sqrt{\frac{\Lambda}{3}}\, t\right)$, goes to zero as $t \to 0$.

What do these three solutions represent? Fact is that they represent the same solution. That is not obvious. $\mathrm{d}s^2 = \mathrm{d}t^2 + a^2(t)\ \mathrm{d}\tilde{s}^2$. Take this and calculate the affine connection and the curvature tensor, and you will find that in all these cases, the space is maximally symmetric.

This is what is called “de Sitter space”.

$$R_{\mu\nu\rho\sigma} \propto g_{\mu\rho}\, g_{\nu\sigma} - g_{\mu\sigma}\, g_{\nu\rho}$$

That tells us that the space is maximally symmetric. Given the value of the scalar curvature, the maximally symmetric space is unique. Embedding in one dimension higher: de Sitter space is embedded in one time, and several spatial directions, and is a hyperboloid.

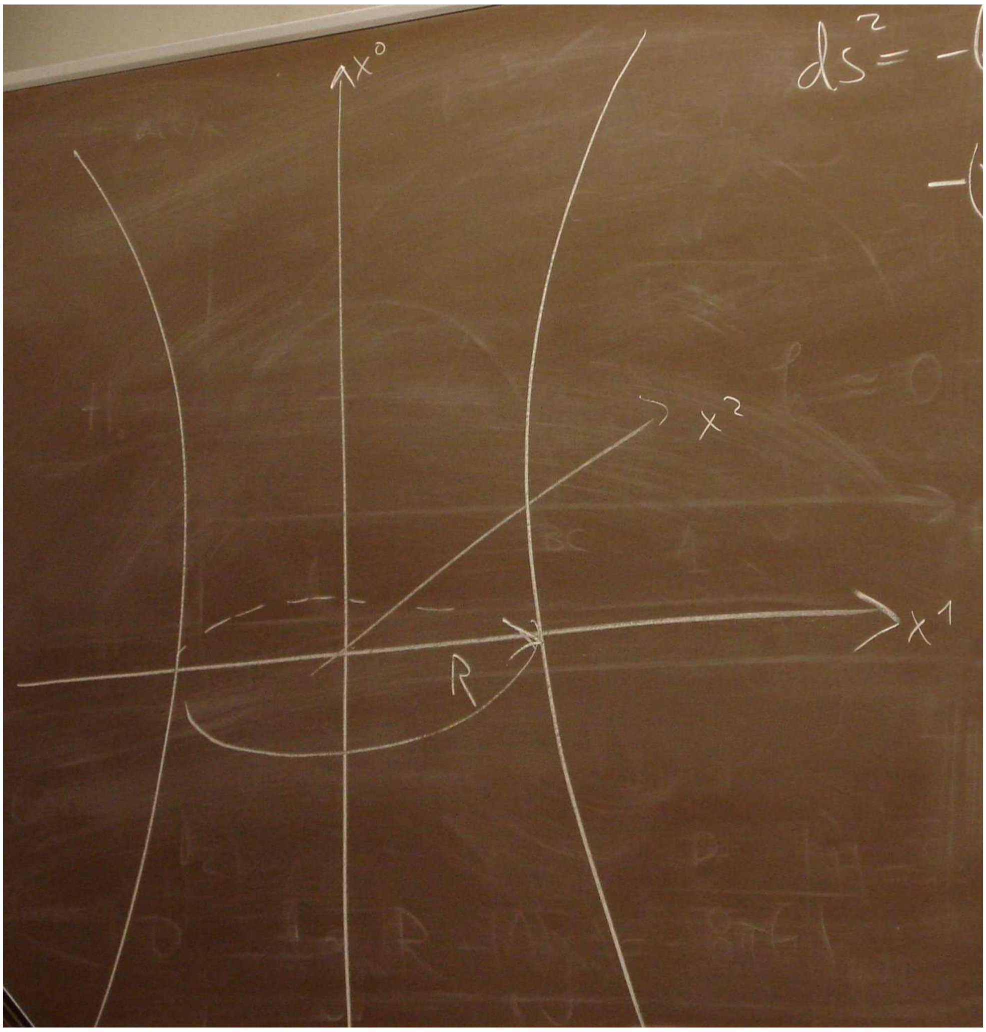

Figure 2. Hyperboloid: de Sitter space. Coordinates $x^0, x^1, ..., x^D$.

$$\mathrm{d}s^2 = -\left(\mathrm{d}x^0\right)^2 + \mathrm{d}x^i \mathrm{d}x^i$$

$$-\left(x^0\right)^2 + x^i x^i = 1 \quad (R^2)$$

1. One way to do this is to say $x^0 = \sinh(t), x^i = \cosh t \cdot \xi^i$, where $|\xi^i| = 1$ (lies on a sphere).

$$\Rightarrow \mathrm{d}s^2 = -\,\mathrm{d}t^2 + \cosh^2 t\, \mathrm{d}\Omega^2$$

This leads directly to the $k = 1$ situation (and then there is some re-scaling, to get the right radius). The other ones are trickier. The first is better, in that this describe the entire de Sitter space. The other ones, $k = 0$ and $k = -1$ only describes parts of de Sitter space.

2. I had to look this up. It is not obvious at all:

$$\begin{cases} x^0 = \frac{1}{2}\left[\mathrm{e}^t - \left(1 + y^2\right)\mathrm{e}^{-t}\right] \\ x^D = \frac{1}{2}\left[\mathrm{e}^t + \left(1 - y^2\right)\mathrm{e}^{-t}\right] \\ x^i = \mathrm{e}^{-t} y^i \qquad\qquad\qquad i, \ldots, D-1 \end{cases}$$

So, de Sitter: maximally symmetric, exponential expansion. That's the important points here.

And that with negative pressure. If you exert pressure (divide the universe into small volumes) when the area moves, it will do work. A positive pressure will do positive work. If there is a pressure, the energy inside will go down. That's what energy conservation says. What it says in this case, since the pressure is negative, the energy inside will increase. It is easy to get very confused about this. A first reaction may be, is there not a violation of conservation of energy here? This is just energy conservation. Because there is negative pressure, when the boundary expands, you get energy instead of loosing it. There is nothing built into energy conservation that makes this unlikely to happen. There is no resistance to creating new energy.

This is the case relevant for cosmology. This is what happens. Not only this, but this plays an important role in the development of the universe.

$\Lambda < 0$: For completeness, let us also consider a negative cosmological constant. $k = -1$ is the only possibility here, since we need a positive $\dot{a}^2$. In this case we get a sine:

$$a = \sqrt{\frac{3}{|\Lambda|}} \sin\left(\sqrt{\frac{|\Lambda|}{3}}\, t\right)$$

It looks like we get a big bang at $t = 0$, and then a big crunch some time later. This is not entirely true. It is because of the choice of coordinates. This is a maximally symmetric space; this is anti-de Sitter space. We visualise this embedded in a space with two times:

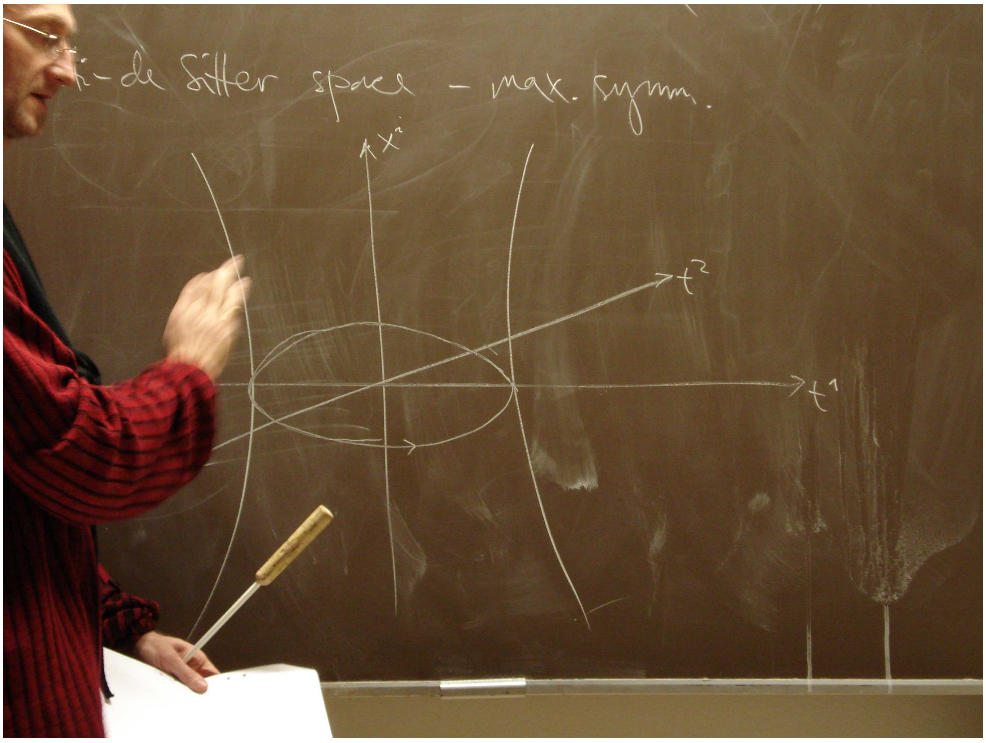

Figure 3. Anti-de Sitter

Circular. But you don't *have* to have a cyclic time. You could visualise it as a roll of toilet paper, where you are on a new sheet after moving one revolution. String theory is full of this kind of spaces, with negative cosmological constant.

At very early times in the universe, Λ is not likely to have been important, so these solutions are not really our universe. But if it stays when the universe expands and becomes dominant, the future of the universe will look like exponential expansion.

In real life, you have different kinds of matter, and we have different values of w. You get some very non-linear equations, and you need to solve them numerically. Thus, in more generality:

$$T_{\mu\nu} = \sum T_{\mu\nu}^{(i)}$$

where i enumerates the different kinds of energy, with different $w^{(i)}$ leading to different $\rho^{(i)}(a)$. There can be a number of complications. Some types of energies may interact with each other. If they are not interacting $T_{\mu\nu}^{(i)}$ are conserved individually, but in general it is only $T_{\mu\nu}$ that is conserved. This becomes especially tricky when you don't have thermal equilibrium: atoms are just forming, radiation is decoupling, ... You might approximate and say that this is a very quick process, and patch solutions in the simpler times. We won't do any of those calculations, at all.

Remember $\Omega_{(i)} = \rho_{(i)}/\rho_{\mathrm{c}}$. Ω_{B} (B for baryonic: atomic nuclei) $\Omega_{\mathrm{B}} \approx 0.04$. But this is not the only type of matter available. There is one specific type of experiment that is very important. There is a particular type of supernova, where a white dwarf eats material from an adjacent star. They always happen in pretty much the same way, with the same luminosity. That enables us to say something about the distance, the redshift and the expansion of the universe. We need more matter. (With matter, I mean something with $w \approx 0$).

$\Omega_{\text{dark matter}} \approx 0.26$. This is not atoms. It is not any known elementary particles. We don't know what it is, but it is a lot more than the matter we see. There are other ways to detect it, e.g. galaxy dynamics. Galaxies move as though there is a lot more matter. If we are really lucky, this could be detected at the LHC.

$\Omega_{\text{radiation}} \approx 0$. Radiation is very small, in this context.

$\Omega_{\text{dark energy}} \approx 0.7$. Something with $w = -1$. This is a mystery. We have no idea what it is. (Or rather, there are lots of *ideas*...) It is something completely covariant (does not break symmetries like the luminiferous ether does).

$\Omega_{\text{total}} \approx 1$.

In the very early universe, there was something like a cosmological constant for a while, making the universe expand exponentially.

Observation (by observing the expansion of the universe) $\Lambda \approx \left(10^{-3}\,\text{eV}\right)^4 \approx 10^{-2}\,\text{J/m}^3$.

In gravity, energy is an absolute concept. In Newtonian mechanics we can shift the energy with a constant, but that does not work here.

Is Λ small or big? Is there anything to compare with?

There is a natural thing to compare with: Planck length, or Planck energy:

$$\text{Planck mass} = \sqrt{\frac{\hbar\, c}{G}} \approx 10^{19}\,\text{GeV} \approx 20\,\mu\text{g}$$

(Don't trust anything when we get near the Planck length. We don't even know what space-time is at that scale.)

This is 10^{30} times larger than the energy scale defined by the cosmological constant. Λ is 10^{30} "too small".

Λ could be a property of the vacuum, but why it is so small is not known. It could be something else.

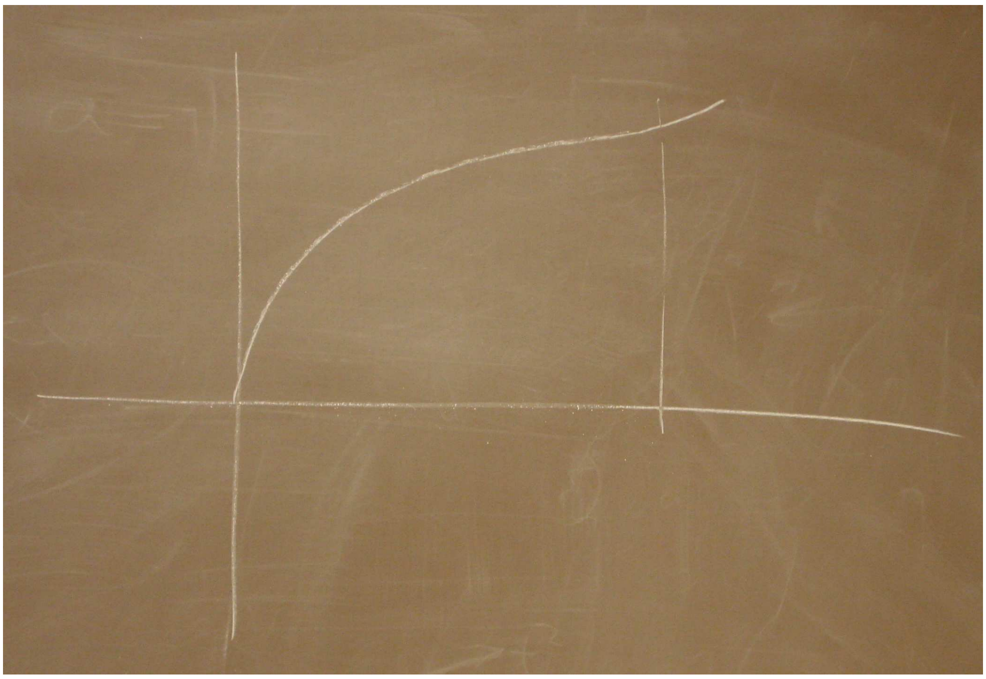

Figure 4. This is our universe. It looks like the cosmological constant is just starting to take over.