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For this part of the course, the cosmology, we use a more modern review linked from the course home page, since Weinberg is a bit outdated.

Ansatz with maximally symmetric space part:

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + a^2(t)\,\tilde{g}_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j, \quad i,j = 1,2,3$$

t is the proper time for an observer at rest. The spatial part can be parametrised with nice coordinates:

$$\tilde{g}_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j = \frac{\mathrm{d}r^2}{1 - k\,r^2} + r^2\,\mathrm{d}\Omega^2$$

where $k \in \{-1, 0, +1\}$. Things at rest will follow geodesics. Sometimes we also use another parametrisation, one where we take the scale factor multiplying time:

$$\mathrm{d}s^2 = a^2(\tau)\big(-\,\mathrm{d}\tau^2 + \tilde{g}_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j\big)$$

where τ is called conformal time. ($\mathrm{d}t = a\,\mathrm{d}\tau$). From Einstein's equations:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3\,p), \quad \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\,\rho - \frac{k}{a^2}$$

where ρ is energy density and p is pressure. We know how they sit in the energy-momentum tensor:

$$T_{\mu\nu} = \left(\begin{array}{c|c} \rho & 0 \\ \hline 0 & p\,\tilde{g}_{ij} \end{array}\right)$$

Energy conservation:

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0$$

We could interpret this as energy and pressure of a spherical volume. Think of a small sphere. This is built into the equations we got directly from Einstein's equations, so we only need two of these three equations.

$p = w\,\rho$. Cold matter (dust): $w = 0$. Dust can be, for instance, galaxies. Galaxies are just like grains of sand, on these scales. Radiation: $w = \frac{1}{3}$.

The energy conservation enables us to solve for the energy density in terms of the scale factor:

$$\rho = \rho_0 \left(\frac{a}{a_0}\right)^{-3(1+w)}, \quad \begin{cases} \text{dust:} & \rho \propto a^{-3} \\ \text{radiation:} & \rho \propto a^{-4} \end{cases}$$

For radiation, we also get a redshift. Today, radiation is essentially negligible in comparison with matter, since the universe has expanded a lot since the time when matter and radiation decoupled, with the formation of the atoms.

Define $H = \dot{a}/a$. This is called the "Hubble constant", but you have to remember that it is, in general, not a constant. It is a name it has. We could call it the "Hubble parameter", but that also sounds a bit like a constant. It is not. The Friedmann equation is now

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}.$$

We can solve it for k:

$$k = a^2\left(\frac{8\pi G}{3}\rho - H^2\right)$$

We can determine H by observation. So if we know how fast the universe expands, and how much matter is in it, we can determine k. There is a critical energy density

$$\rho_{\mathrm{c}} = \frac{3\,H^2}{8\pi G}$$

$$\begin{array}{lcl} \rho > \rho_{\mathrm{c}} & \Rightarrow & k = 1 \\ \rho = \rho_{\mathrm{c}} & \Rightarrow & k = 0 \\ \rho < \rho_{\mathrm{c}} & \Rightarrow & k = -1 \end{array}$$

A very convenient measure of how much energy there is in the universe is $\Omega_{(i)} := \rho_{(i)}/\rho_{\mathrm{c}}$, the energy density compared to the critical density. We can do this for any type of energy, labelled by index (i).

Solve when $k=0$, and one specific value of w. The Friedmann equation:

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho_0\left(\frac{a}{a_0}\right)^{-3(1+w)}$$

$$\Rightarrow \quad \dot{a} = \text{const} \times a^{-\frac{3}{2}(1+w)+1}$$

This is a separable, first order differential equation:

$$\mathrm{d}a\, a^{\frac{3}{2}(1+w)-1} = \text{const} \times \mathrm{d}t$$

$$\frac{t}{t_0}(+\cdots) = a^{\frac{3}{2}(1+w)}$$

Solving for $a(t)$:

$$a(t) = a_0\left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}}$$

This is for $w \neq -1$. For $w = -1$ we get instead $a(t) = a_0\,\mathrm{e}^{Ht}$. (When this is the case, the Hubble constant H actually *is* a constant.)

As long as $w > -1$ (which is clearly the case for dust and radiation, and any sort of thing that I guess you can imagine)... As long as $w > -1$ the universe exhibits a “Big Bang” at $t=0$. This is a true singularity, a curvature singularity. There is a singularity in the theory, but we have the problem that we don’t really know how things behave at extremely high energies or extremely small scales, so we can’t say for sure that there is a singularity in the real world.

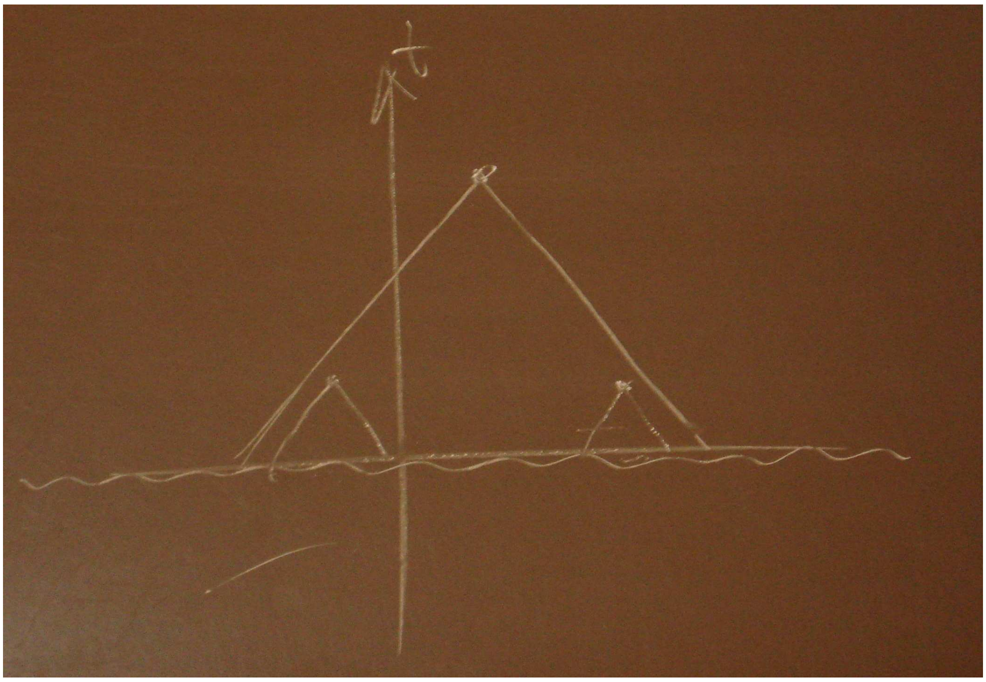

Figure 1. The wavy part is the singularity. Note the difference from the Schwarzschild solution. The triangles are light cones. Two points far back in time have no causal connection, but we can nevertheless see that the universe looks pretty much the same in every direction. (We would be the observer in the topmost triangle, in this figure.)

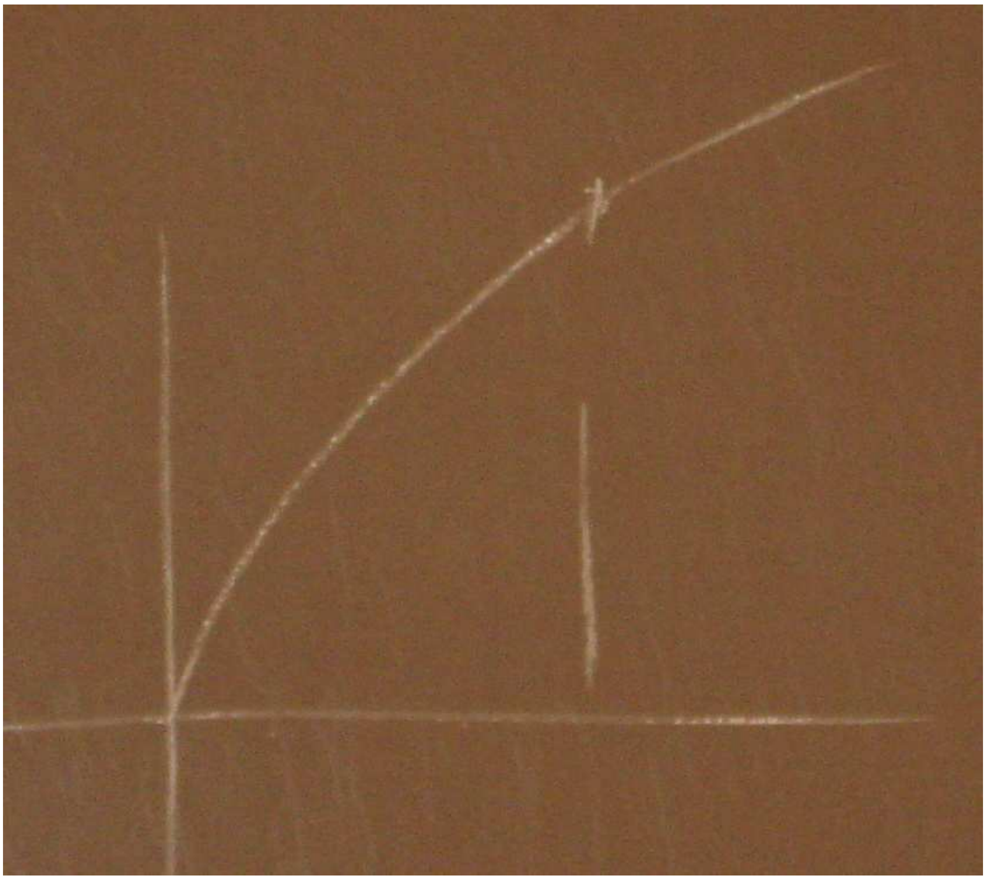

Figure 2. There is no re-collapse here. $a(t)$ grows all the time. See also figure 3, the $k=0$ case.

Let us estimate t_0: The age of the universe. (We are thinking of t_0 as the present time, and a_0 as the present scale factor.)

$$H = \frac{\dot{a}}{a} = \frac{2}{3(1+w)\,t}$$

Astronomers use $H_0 = 100\ h$ km/(s Mpc). The astronomers give the dimensionless number h. Observations now, is $h \approx 0.71 \pm 0.06$.

$$\begin{gathered} 1\,\mathrm{pc} \approx 3.26\ \mathrm{ly} \\ 1\,\mathrm{ly} \approx 9.46 \times 10^{12}\,\mathrm{km} \\ 1\,\mathrm{y} \approx 3.15 \times 10^{7}\,\mathrm{s} \end{gathered}$$

$$H_0 = \frac{1}{4.34 \times 10^{17}\,\mathrm{s}} \approx \frac{1}{13.8 \times 10^{9}\ \mathrm{y}}$$

This gives a rough estimate of the age of the universe.

Non-flat universes: It turns out that conformal time τ is a more suitable parameter.

$$\tau: \quad a\,\mathrm{d}t = \mathrm{d}\tau$$

We have to member that age is not measured in τ, so we have to convert to measured time, t.

$$\dot{f} \equiv \frac{\mathrm{d}f}{\mathrm{d}t} = \frac{1}{a}\frac{\mathrm{d}f}{\mathrm{d}\tau} \equiv \frac{1}{a} f'$$

$$\dot{a} = \frac{a'}{a} \equiv h$$

(Not to be confused with the astronomer's h above. We forget about that for the moment.)

$$\ddot{a} = \frac{h'}{a}$$

We use both equations we had from Einstein's equations. The acceleration equation:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3\,p) \quad \Rightarrow \quad h' = -\frac{4\pi G}{3}(\rho + 3\,p)\,a^2 = \frac{4\pi G}{3}(1 + 3w)\,\rho\,a^2$$

The other equation:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \quad \Rightarrow \quad \frac{h^2}{a^2} = \frac{8\pi G}{3}\,\rho - \frac{k}{a^2} \quad \Rightarrow h^2 = \frac{8\pi G}{3}\,\rho\,a^2 - k$$

We want to eliminate a. This is something we can do with this time parameter, but not with the other one.

$$h' = -\frac{1 + 3\,w}{2}\left(h^2 + k\right)$$

Separable:

$$\frac{\mathrm{d}h}{h^2 + k} = -\frac{1 + 3\,w}{2}\,\mathrm{d}\tau$$

$$-\frac{1 + 3w}{2}\tau = \begin{cases} \tan^{-1}h + \mathrm{const} = -\cot^{-1}h & \text{if } k = 1 \\ -\frac{1}{h} & \text{if } k = 0 \\ -\coth^{-1}h & \text{if } k = -1 \end{cases}$$

$$h(\tau) = \begin{cases} \cot\!\left(\frac{1 + 3w}{2}\tau\right) & \text{if } k = 1 \\ \frac{2}{1 + 3w}\tau^{-1} & \text{if } k = 0 \\ \coth\!\left(\frac{1 + 3w}{2}\tau\right) & \text{if } k = -1 \end{cases}$$

$$h^2 + k = \begin{cases} \frac{1}{\sin^2\left(\frac{1 + 3w}{2}\tau\right)} \\ \frac{1}{(...)^2} \\ \frac{1}{\sinh^2(...)} \end{cases}$$

$$h^2 + k = \frac{8\pi G}{3}\,\rho\,a^2 \propto a^{-(1 + 3w)}$$

$$a(\tau) \propto \left\{ \begin{array}{ll|ll|ll} \left[\sin\!\left(\frac{1 + 3w}{2}\tau\right)\right]^{\frac{2}{1 + 3w}} & \text{if } k = 1 & \text{dust:} & \sin^2\!\left(\frac{\tau}{2}\right) = \frac{1}{2}(1 - \cos\tau) & \text{radiation:} & \sin\tau \\ t^{\frac{2}{1 + 3w}} & \text{if } k = 0 & & \tau^2 & & \tau \\ \left[\sinh\!\left(\frac{1 + 3w}{2}\tau\right)\right]^{\frac{2}{1 + 3w}} & \text{if } k = -1 & & \frac{1}{2}(\cosh\tau - 1) & & \sinh\tau \end{array} \right.$$

If we want to say something about the universe, we have to go back to the time variable t. We have $a\,\mathrm{d}t = \mathrm{d}\tau$

		dust	radiation
$k=1$:	$t(\tau)=$	$\tau-\sin\tau$	$1-\cos\tau$
$k=0$:	$t(\tau)=$	τ^3	τ^2
$k=-1$:	$t(\tau)=$	$\sinh\tau-\tau$	$\cosh\tau-1$

What we see here is something interesting. We see that for $k=+1$, a is periodic. We have a periodic universe when it is closed. When it is open, infinite, it goes on expanding.

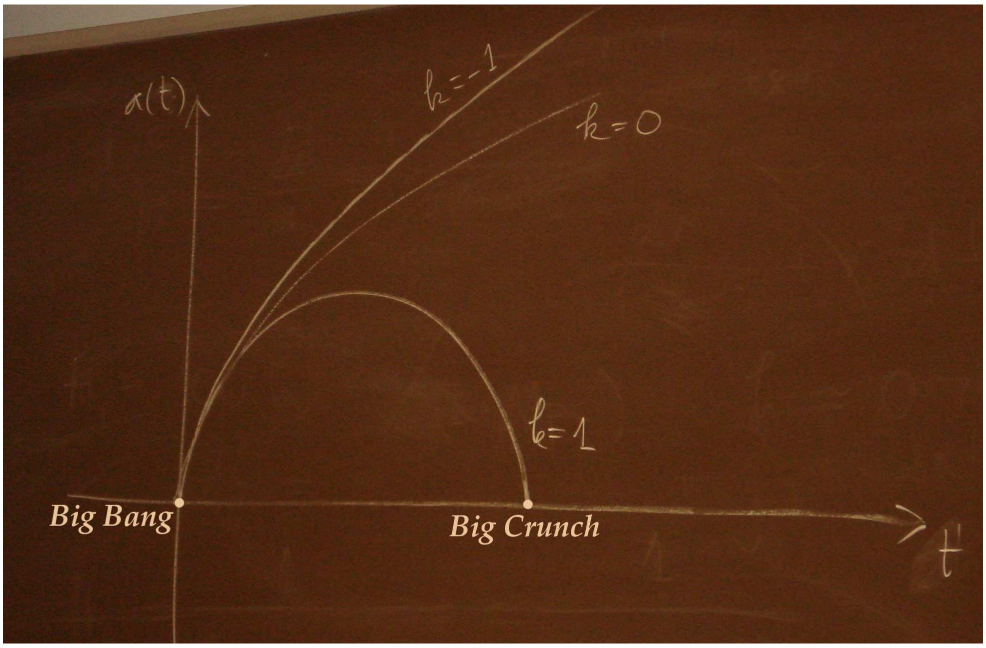

Figure 3. a as a function of t for $k=+1,0,-1$.

Small times: $a\propto\tau^{2/(1+3w)}$. Assume $\frac{2}{1+3w}>0$. (If we want to talk about the cosmological constant, this does not hold.)

$$a\propto\tau^{2/(1+3w)},\quad t\propto\tau^{\frac{3(1+w)}{1+3w}}\quad\Rightarrow\quad a\propto t^{\frac{2}{3(1+w)}}$$

All common matter has a Big Bang, but only $k=1$ has a Big Crunch.