2008–12–04

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Today: Symmetric spaces and Cosmology.

7.1

> "Find all Killing vectors on the two-sphere embedded in $\mathbb{R}^3$."

We assume we are on the unit two-sphere S^2. From before:

$$\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2$$

$$\Gamma^\theta_{\varphi\varphi} = -\cos\theta\sin\theta, \quad \Gamma^\varphi_{\theta\varphi} = \cot\theta$$

Killing vectors are the generators of *isometries*, i.e. transformations $x^\mu \to x^\mu + \varepsilon\,\xi^\mu$ where ε is an infinitesimal parameter, and ξ^μ is the killing vector; transformations that preserve the *form* of the metric.

$$g_{\mu\nu}(x) = \frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu}\, g_{\rho\sigma}(x') \tag{1}$$

The same function g on both sides.

Finding all isometries of the space is equivalent to finding all the Killing vectors ξ^μ. The condition (1) yields the Killing equation $\mathrm{D}_{(\mu}\xi_{\nu)} = \mathrm{D}_\mu\xi_\nu + \mathrm{D}_\nu\xi_\mu = 0$. Let's solve the Killing equation for the two-sphere. In our case $\xi^\mu = \xi^\mu(\theta,\varphi)$.

$\mu = \nu = \theta$:

$$\mathrm{D}_\theta\,\xi_\theta + \mathrm{D}_\theta\xi_\theta = 2\big(\partial_\theta\xi_\theta - \Gamma^\lambda_{\theta\theta}\,\xi_\lambda\big) = 2\partial_\theta\xi_\theta = 0$$

$$\Rightarrow \quad \partial_\theta\xi_\theta = 0 \tag{2}$$

$\mu = \theta, \nu = \varphi$:

$$\mathrm{D}_\theta\xi_\varphi + \mathrm{D}_\varphi\xi_\theta = \partial_\theta\,\xi_\varphi - \underbrace{\Gamma^\rho_{\theta\varphi}\xi_\rho}_{=\Gamma^\varphi_{\theta\varphi}\xi_\varphi} + \partial_\varphi\xi_\theta - \underbrace{\Gamma^\rho_{\varphi\theta}\xi_\rho}_{=\Gamma^\varphi_{\varphi\theta}\xi_\varphi} = 0$$

$$\Rightarrow \quad \partial_\theta\xi_\varphi + \partial_\varphi\xi_\theta = 2\cot\theta\,\xi_\varphi \tag{3}$$

$\mu = \nu = \varphi$:

$$2\,\mathrm{D}_\varphi\,\xi_\varphi = 2\left(\partial_\varphi\xi_\varphi - \Gamma^\rho_{\varphi\varphi}\,\xi_\rho\right) = 2\left(\partial_\varphi\xi_\varphi - \Gamma^\theta_{\varphi\varphi}\xi_\theta\right) = 0$$

$$\Rightarrow \quad \partial_\varphi\xi_\varphi = -\cos\theta\sin\theta\cdot\xi_\theta \tag{4}$$

$(2) \Rightarrow \xi_\theta = f(\varphi)$, i.e. ξ_θ is independent of θ. Insert in (4) $\Rightarrow$

$$\partial_\varphi\xi_\varphi = -\cos\theta\sin\theta\, f(\varphi)$$

Integrate with respect to φ:

$$\Rightarrow \quad \xi_\varphi = -\cos\theta\sin\theta\underbrace{\int_{\varphi_0}^{\varphi}\mathrm{d}\varphi'\, f(\varphi')}_{=:F(\varphi)} + g(\theta)$$

Insert in (3) $\Rightarrow$

$$\left(\sin^2\theta - \cos^2\theta\right) F(\varphi) + \frac{\mathrm{d}g(\theta)}{\mathrm{d}\theta} + \frac{\mathrm{d}f(\varphi)}{\mathrm{d}\varphi} = 2\,\frac{\cos\theta}{\sin\theta}\,(-\cos\theta\sin\theta\,F(\varphi) + g(\theta))$$

$$= -2\cos^2\theta\,F(\varphi) + 2\,\frac{\cos\theta}{\sin\theta}\,g(\theta)$$

Using the Pythagorean trigonometric identity:

$$\Rightarrow F(\varphi) + \frac{\mathrm{d}f(\varphi)}{\mathrm{d}\varphi} = -\frac{\mathrm{d}g(\theta)}{\mathrm{d}\theta} + 2\,\frac{\cos\theta}{\sin\theta}\,g(\theta)$$

The equation is separable. The left hand side does not depend on θ, and the right hand side does not depend on φ. Thus they both equal a constant:

$$\frac{\mathrm{d}g(\theta)}{\mathrm{d}\theta} - 2\,\frac{\cos\theta}{\sin\theta}\,g(\theta) = C_1 \tag{5}$$

$$F(\varphi) + \frac{\mathrm{d}f(\varphi)}{\mathrm{d}\varphi} = -C_1 \tag{6}$$

(5) $\Rightarrow$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{g(\theta)}{\sin^2\theta}\right) = \frac{C_1}{\sin^2\theta}$$

Integrate with respect to θ:

$$g(\theta) = \sin^2\theta\,(-C_1\cot\theta + C)$$

Differentiate (6) with respect to φ $\Rightarrow$

$$f(\varphi) + \frac{\mathrm{d}^2 f(\varphi)}{\mathrm{d}\varphi^2} = 0$$

This is the wave equation.

$$f(\varphi) = A\sin\varphi + B\cos\varphi$$

Integrate:

$$F(\varphi) = -A\cos\varphi + B\sin\varphi + C_2$$

Insert into (6) $\Rightarrow C_2 = -C_1$.

The solution to the Killing equation is

$$\xi_\theta = A\sin\varphi + B\cos\varphi$$

$$\xi_\varphi = -\cos\theta\sin\theta(-A\cos\varphi + B\sin\varphi - C_1) + \sin^2\theta\left(-C_1\,\frac{\cos\theta}{\sin\theta} + C\right) =$$

$$= \cos\theta\sin\theta(A\cos\varphi - B\sin\varphi) + C\sin^2\theta$$

Three independent parameters in the general solution: A, B and C. There are three independent basis vectors in the space of solutions of $\mathrm{D}_{(\mu}\xi_{\nu)} = 0$. In other words, there are three independent basis vectors in the space of isometries. These Killing vectors correspond to rotations around the x, y and z axis, respectively, i.e. *all* possible rotations in the embedding space $\mathbb{R}^3$.

In general, the maximal number of Killing vectors (isometries) in N-dimensional space is $N(N+1)/2$. Here, $N=2 \Rightarrow 2(2+1)/2=3$. S^2 is maximally symmetric!

7.2

> "Consider the Robertson-Walker metric
>
> $$\mathrm{d}\tau^2=\mathrm{d}t^2-R^2(t)\left[\frac{\mathrm{d}r^2}{1-k r^2}+r^2\,\mathrm{d}\theta^2+r^2\sin^2\theta\,\mathrm{d}\varphi^2\right].$$
>
> Investigate the properties of the spatial part in the three cases $k=1, k=0$ and $k=-1$."

• $k=1$. We reparametrise this, out of the blue, as $r=\sin\psi$, $\mathrm{d}r=\cos\psi\,\mathrm{d}\psi$:

$$\Rightarrow \frac{\mathrm{d}r^2}{1-r^2}=\frac{\cos^2\psi\,\mathrm{d}\psi^2}{1-\sin^2\psi}=\mathrm{d}\psi^2$$

$$\Rightarrow \mathrm{d}\tau^2=\mathrm{d}t^2-R^2(t)\left[\mathrm{d}\psi^2+\sin^2\psi\,\mathrm{d}\theta^2+\sin^2\psi\sin^2\theta\,\mathrm{d}\varphi^2\right]$$

Compare to the three-sphere S^3 embedded in $\mathbb{R}^4$:

$$x_1^2+x_2^2+x_3^2+x_4^2=1 \quad \text{(unit 3-sphere)}$$

To parametrise S^3 we use hyperspherical coordinates:

$$\begin{cases} x_1=\sin\psi\sin\theta\cos\varphi \\ x_2=\sin\psi\sin\theta\sin\varphi \\ x_3=\sin\psi\cos\theta \\ x_4=\cos\psi \end{cases}, \quad \begin{cases} 0\leqslant\varphi\leqslant 2\pi \\ 0\leqslant\psi\leqslant\pi \\ 0\leqslant\psi\leqslant\pi \end{cases}$$

$$\mathrm{d}x_1=\ldots,\quad \mathrm{d}x_2=\ldots,\quad \mathrm{d}x_3=\ldots,\quad \mathrm{d}x_4=\ldots.$$

$$\mathrm{d}s^2=\mathrm{d}x_1^2+\mathrm{d}x_2^2+\mathrm{d}x_3^2+\mathrm{d}x_4^2\big|_{x_1^2+x_2^2+x_3^2+x_4^2=1}=\mathrm{d}\psi^2+\sin^2\psi\,\mathrm{d}\theta^2+\sin^2\psi\sin^2\theta\,\mathrm{d}\varphi^2$$

For $k=1$ the spatial part of the metric describes a three-sphere of radius $R(t)$ embedded in $\mathbb{R}^4$.

•$k=0$.

$$\mathrm{d}\tau^2=\mathrm{d}t^2+R^2(t)\left[\mathrm{d}r^2+r^2\mathrm{d}\theta^2+r^2\sin^2\theta\,\mathrm{d}\varphi^2\right]$$

This is flat 3D Euclidean space in spherical coordinates. The spatial part is flat but the relation between the spatial and temporal parts scale with $R(t)$.

•$k=-1$. $r=\sinh\psi$, $\mathrm{d}r=\cosh\psi\,\mathrm{d}\psi$

$$\Rightarrow \quad \frac{\mathrm{d}r^2}{1-k r^2}=\frac{\cosh^2\psi\,\mathrm{d}\psi^2}{1+\sinh^2\psi}=\mathrm{d}\psi^2$$

$$\Rightarrow \quad \mathrm{d}\tau^2=\mathrm{d}t^2-R^2(t)\left[\mathrm{d}\psi^2+\sinh^2\psi\,\mathrm{d}\theta^2+\sinh^2\psi\sin^2\theta\,\mathrm{d}\varphi^2\right]$$

In this case the spatial part *cannot* be embedded in $\mathbb{R}^4$. However, it *can* be embedded in four-dimensional Minkowski space $\mathbb{R}^{3,1}$. Consider a three-dimensional hyperboloid in 4-dimensional Minkowski space:

$$x_1^2+x_2^2+x_3^2-x_4^2=-1$$

This one means that we have a unit "radius of curvature". Parametrise (compare to hyperspherical):

$$\begin{cases} x_1 = \sinh\psi\sin\theta\cos\varphi \\ x_2 = \sinh\psi\sin\theta\sin\varphi \\ x_3 = \sinh\psi\cos\theta \\ x_4 = \cosh\psi \end{cases}$$

$$\Rightarrow \quad \mathrm{d}s^2 = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 - \mathrm{d}x_4^2|_{x_1^2+x_2^2+x_3^2+x_4^2=-1} = \mathrm{d}\psi^2 + \sinh^2\psi\,\mathrm{d}\theta^2 + \sinh^2\psi\sin^2\theta\,\mathrm{d}\varphi^2$$

The spatial part for $k = -1$ describes a 3-dimensional hyperboloid, embedded in $\mathbb{R}^{3,1}$ with "radius of curvature" $R(t)$. Anti de Sitter (AdS_3) space.

To summarise:

$k = 1$: closed space (finite, unbounded)

$k = 0, -1$: open space (infinite, unbounded)