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Plan: Einstein's equations for the universe. What we need is:

- Ansatz for the metric. Observations of the universe: the spatial part is homogeneous and isotropic — maximally symmetric at a given time (for a given time coordinate). This leads us to the left hand side of Einstein's equations.

- Models for matter and energy. This is needed for the right hand side.

Metric by embedding: Embed the space in a space with one more dimension. If we do that in a nice way, we know that we have all of the symmetry. Consider $(D+1)$-dimensional space (or spacetime) with $\mathrm{d}s^2 = \ldots$ The coordinates we want to use are x^μ, $\mu = 0, \ldots, D-1$ and $x^D = z$. We are going to eliminate z at the end. $\mathrm{d}s^2 = C_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu + \ldots$ for some constant matrix $C_{\mu\nu}$. (We can always change coordinates to any such matrix with the same sign signature.) $\mathrm{d}s^2 = C_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu + \frac{1}{K}\,\mathrm{d}z^2$. This is just a flat space. In order to get down to the D-dimensional space, we restrict to a surface where $C_{\mu\nu}\,x^\mu\,x^\nu + \frac{1}{K}\,z^2 = \frac{1}{K}$. This is a sphere or hyperboloid, of some kind. This K happens to be the same K that was mentioned yesterday: minus the scalar curvature.

We want to get rid of z, so we take

$$K\,C_{\mu\nu}\,x^\mu\,x^\nu + z^2 = 1$$

and differentiate:

$$K\,C_{\mu\nu}\,x^\mu\,\mathrm{d}x^\nu + z\,\mathrm{d}z = 0$$

($C_{\mu\nu}$ is a symmetric matrix, of course.) This enables us to eliminate z. To simplify notation, let us define $\mathrm{d}x \cdot \mathrm{d}x \equiv C_{\mu\nu}\, \mathrm{d}x^{\mu}\, \mathrm{d}x^{\nu}$.

$$K\, x \cdot \mathrm{d}x + z\, \mathrm{d}z = 0$$

$$\mathrm{d}z^2 = \frac{K^2 \big(x \cdot \mathrm{d}x^2\big)}{z^2} = \frac{K^2 (x \cdot \mathrm{d}x)^2}{1 - K\, x^2}, \quad x^2 \text{ meaning } x \cdot x, \text{ in the same notation.}$$

$$\mathrm{d}s^2 = \mathrm{d}x \cdot \mathrm{d}x + K\, \frac{(x \cdot \mathrm{d}x)^2}{1 - K\, x \cdot x}$$

Choose $C_{\mu\nu} = |K|^{-1}\, \eta_{\mu\nu}$ where $\eta_{\mu\nu}$ is a diagonal matrix with only ± 1.

$$\mathrm{d}s^2 = |K|^{-1} \left(\mathrm{d}x^2 + k\, \frac{x \cdot \mathrm{d}x}{1 - k\, x^2} \right), \quad \text{where } \mathrm{d}x^2 \text{ is taken with } \eta_{\mu\nu} \text{ and } \mathrm{sign}(K) = k \in \{+1, -1, 0\}.$$

Every scalar product above is taken with $\eta_{\mu\nu}$: $x^2 = x^{\mu}\, x^{\nu}\, \eta_{\mu\nu}$

For $\eta = \mathbf{1}$:

$$\begin{cases} k > 0\text{:} & \text{we have a sphere} \quad (0, D+1) \to (0, D) \quad (\text{time dimensions, space dimensions}) \\ k < 0\text{:} & \text{hyperbolic space} \quad (1, D) \to (0, D) \end{cases}$$

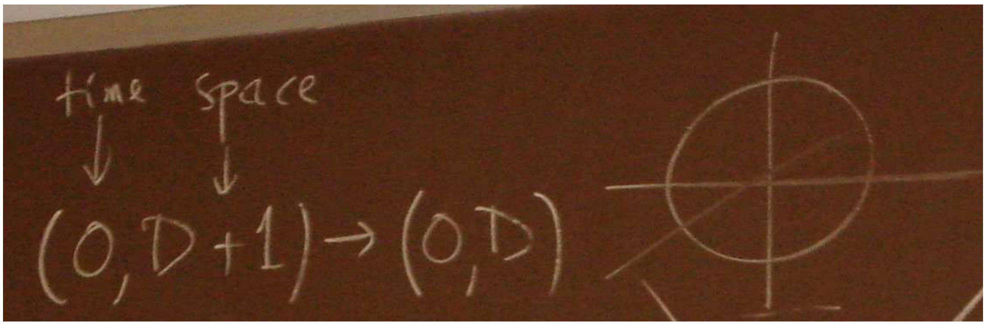

Figure 1. $\eta = \mathbf{1}$, $k > 0$. This is a sphere.

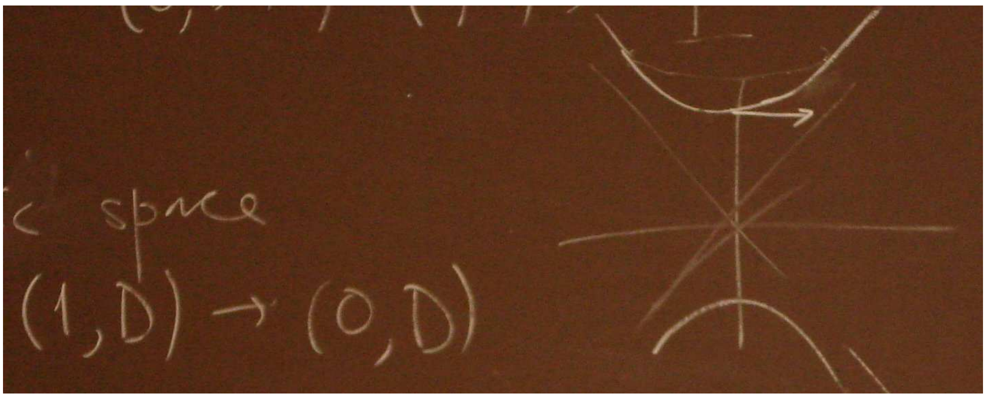

Figure 2. $\eta = \mathbf{1}, k < 0$. This is a hyperbolic space, with embedding $(1, D) \to (0, D)$.

For $\eta = \mathrm{diag}(-1, 1, 1, 1)$.

$$\begin{cases} k > 0\colon & \text{de Sitter} & (1, D) \rightarrow (1, D-1) \\ k < 0\colon & \text{anti-de Sitter} & (2, D-1) \rightarrow (1, D-1) \end{cases}$$

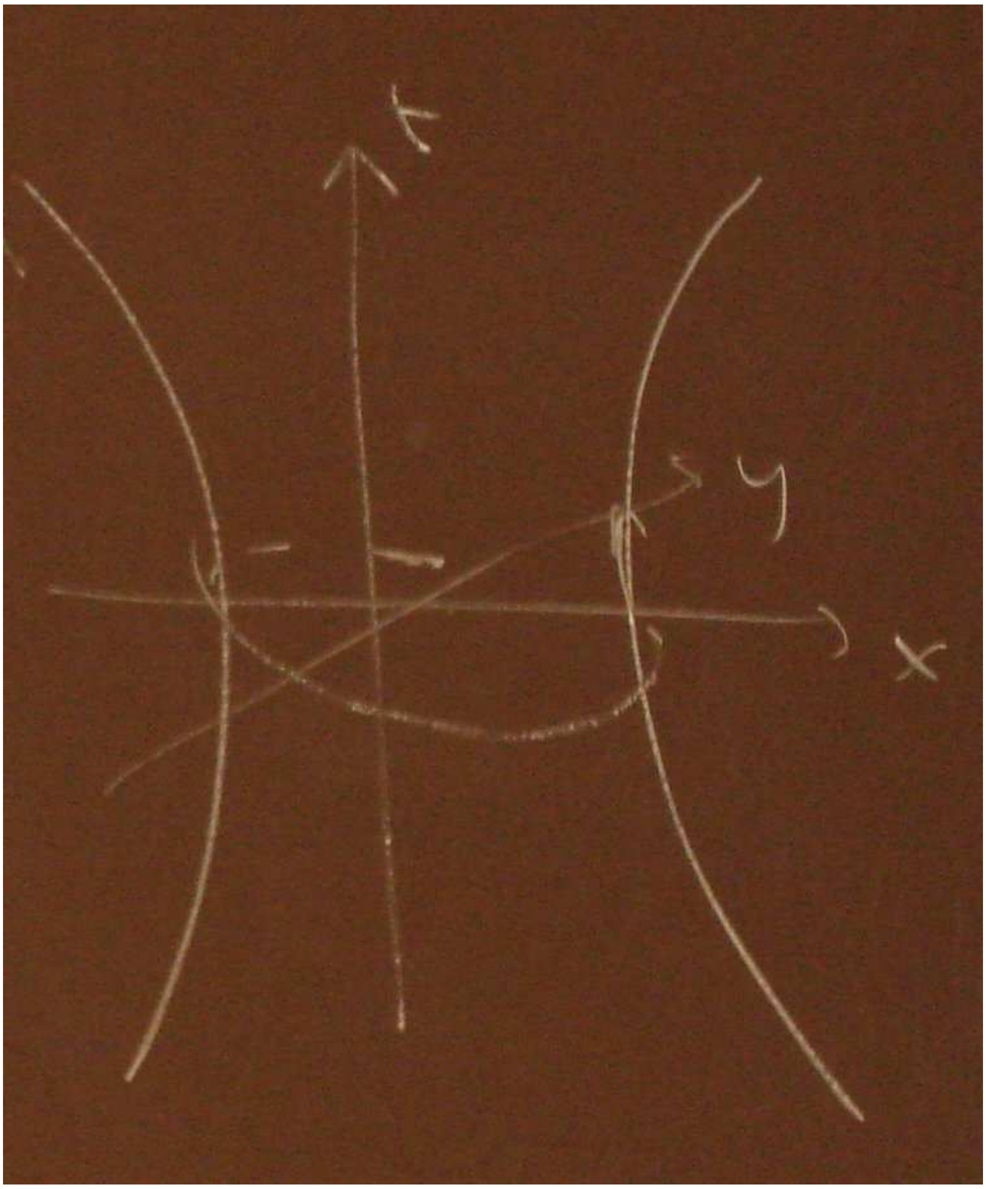

Figure 3. This is de Sitter.

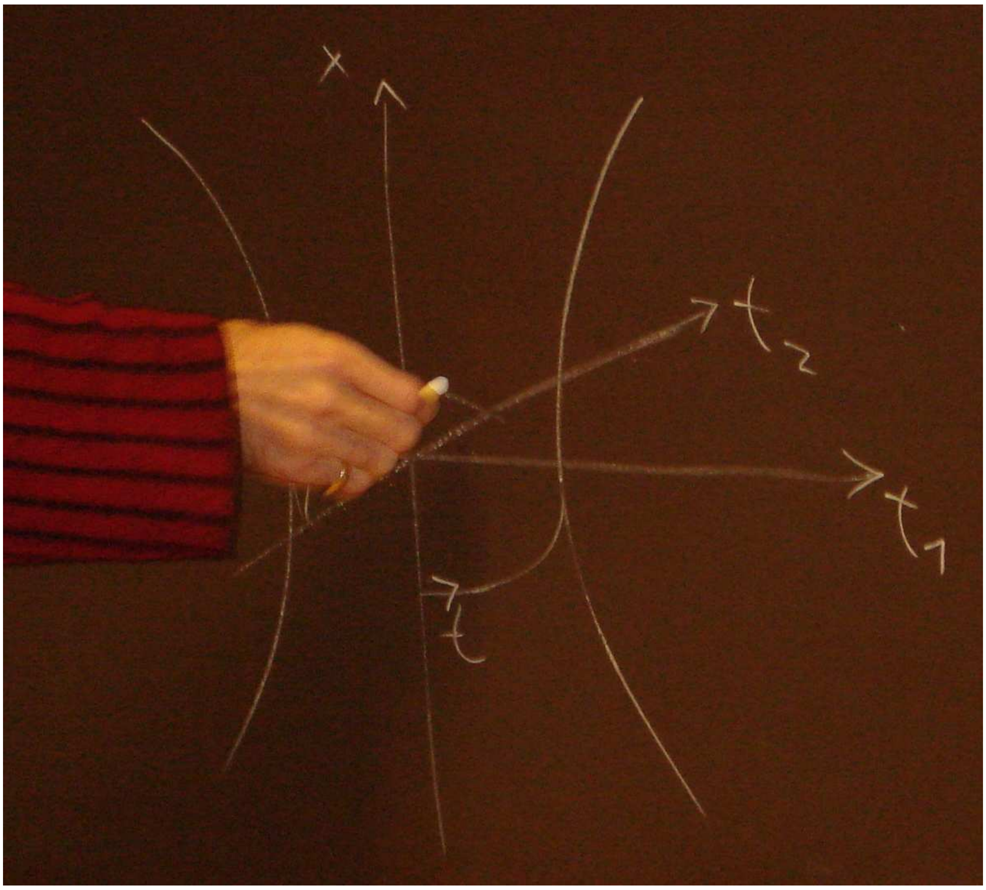

Figure 4. Anti-de Sitter. Note that it is embedded in a space with two time directions.

For cosmology: The spatial part of the metric should be something like this:

$$\mathrm{d}s^2 = |K|^{-1}\left(\mathrm{d}x^2 + k\,\frac{(x\cdot\mathrm{d}x)^2}{1-k\,x\cdot x}\right)$$

where $|K|^{-1}$ is just some normalisation factor (it will become time dependent, once we start doing cosmology). Euclidean signature, $D=3$.

$$\mathrm{d}x^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2$$

$$x\cdot\mathrm{d}x = r\,\mathrm{d}r$$

$$\mathrm{d}s^2 = |K|^{-1}\left(\mathrm{d}r^2 + \frac{k\,r^2\,\mathrm{d}r^2}{1-k\,r^2} + r^2\mathrm{d}\Omega^2\right) = |K|^{-1}\left(\frac{\mathrm{d}r^2}{1-k\,r^2} + r^2\,\mathrm{d}\Omega^2\right)$$

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + a^2(t)\left(\frac{\mathrm{d}r^2}{1-k\,r^2} + r^2\,\mathrm{d}\Omega^2\right)$$

If the universe is a sphere a would be the radius. Otherwise, it is just a scale factor.

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + a^2(t)\,\tilde{g}_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$$

Any function in front of t could always be absorbed into $a^2(t)$ by a coordinate transformation, so this will be enough for our ansatz. t is the proper time for an observer at rest.

Sometimes: $\mathrm{d}s^2 = a^2(\tau)\big[-\,\mathrm{d}\tau^2 + \tilde{g}_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j\big]$. τ is not proper time. τ is "conformal time".

Affine connection:

$$\Gamma^0_{00} = 0, \quad \Gamma^0_{0i} = 0, \quad \Gamma^i_{00} = 0$$

Constant x^i is a geodesic.

$$\Gamma^0_{ij} = a\,\dot{a}\,\tilde{g}_{ij}, \quad \Gamma^i_{0j} = \frac{\dot{a}}{a}\,\delta^i{}_j\,, \quad \Gamma^i_{jk} = \tilde{\Gamma}^i_{jk}$$

Ricci ["You should do this yourself"]. Now for three spatial dimensions:

$$R_{00} = 3\,\frac{\ddot{a}}{a}, \quad R_{ij} = -\,(a\,\ddot{a} + 2\,\dot{a}^2 + 2\,k)\tilde{g}_{ij}$$

Use

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\,R = -\,8\,\pi\,G\,T_{\mu\nu}$$

or equivalently

$$R_{\mu\nu} = -\,8\pi\,G\bigg(T_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}\,T^\lambda{}_\lambda\bigg)$$

$T_{\mu\nu} = ?$

$T_{00} = \rho =$ energy density. $T_{0i} = 0$. We expect this to be zero. There is no natural vector where it could point. We have a symmetry in the metric. $T_{ij} = p\,g_{ij} = p\,a^2\,\tilde{g}_{ij}$. For the moment, p is just a letter. But we call it pressure. (We have to do it this way, because we did not go through the hydrodynamics.) Thus, the ansatz we want to use:

$$\begin{cases} T_{00} = \rho, \qquad T_{0i} = 0 \\ T_{ij} = p\,g_{ij} = p\,a^2\tilde{g}_{ij} \\ T^\mu{}_\mu = -\,\rho + 3\,p \end{cases}$$

• "Dust" (i.e. "cold" matter, matter at low velocities):

$$T_{\mu\nu} \propto P_\mu\,P_\nu$$

where P_μ is the momentum of the particles. If they are at rest, we have

$$T_{\mu\nu} \doteq \left(\begin{array}{c|ccc} \rho & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \text{ for dust}$$

$p = 0$ for dust.

• Radiation: Electromagnetism has $T^\lambda{}_\lambda = 0$. This means something, but I am not going to talk about it. This gives us $p = \frac{1}{3}\,\rho$. When we look at energy conservation, this has a very natural interpretation.

Generic situation for dust, radiation and maybe some other types of energy too, we have $p = w\,\rho$ for some constant w.

$$\text{rhs} \propto T_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} T^{\lambda}{}_{\lambda}$$

$$3\,\frac{\ddot{a}}{a} = -\,4\pi\, G\,(\rho + 3p)\text{: acceleration equation}$$

$$a\ddot{a} + 2\dot{a}^2 + 2k = 4\pi G\, a^2(\rho - p)$$

Insert the acceleration equation into the later equation

$$\dot{a}^2 + k = \frac{8\pi\, G}{3}\, a^2\, \rho\text{:}\quad \text{the Friedmann equation}$$

Energy conservation we get from the zeroth component of $\mathrm{D}_\mu T^{\mu\nu} = 0$. (It gives us no extra information, it is built into the Einstein equations. But it *may* give us a simpler equation to replace the acceleration or the Friedmann equation.) The i-components are empty: check!

$$0 = g^{\nu\lambda}\mathrm{D}_\nu T_{\lambda 0} = g^{00}\mathrm{D}_0 T_{00} + g^{ij}\mathrm{D}_i T_{j0} =$$

Beware: $T_{j0} = 0$ does *not* imply that $\mathrm{D}_i T_{j0} = 0$. T_{j0} is just one corner of a tensor, and the affine connection in D_i may mix different parts of the tensor.

$$= g^{00}\Big(\partial_0 T_{00} - 2\cancel{\Gamma^0_{00}}T_{00} - 2\,\cancel{\Gamma^i_{00}}T_{0i}\Big) + g^{ij}\Big(\partial_i \cancel{T_{j0}} - \Gamma^0_{ij}\,T_{00} - \Gamma^k_{ij}\cancel{T_{k0}} - \Gamma^0_{i0}\cancel{T_{j0}} - \Gamma^k_{i0}T_{jk}\Big) =$$

$$= -\,\dot{\rho} - 3\,\frac{\dot{a}}{a}(\rho + p)$$

$$0 = \underbrace{\dot{\rho}\, a^3 + 3\, a^2\, \dot{a}\rho}_{=\frac{\mathrm{d}}{\mathrm{d}t}(\rho a^3)} + 3\, a^2 \dot{a} p$$

$$\mathrm{d}\big(\rho a^3\big) = -\,3\, p\, a^2\, \mathrm{d}a$$

$$\mathrm{d}\left(\frac{4\pi a^3}{3}\,\rho\right) = -\,4\pi\, a^2\, p\, \mathrm{d}a$$

Volume times ρ equals energy. Area times p equals force.

Use conservation of energy together with Friedmann.

$$\frac{\mathrm{d}}{\mathrm{d}a}\big(\rho\, a^3\big) = -\,3\, p\, a^2 = -\,3w\rho\, a^2$$

$$\rho(a) \propto a^{\alpha}$$

$$\frac{\mathrm{d}}{\mathrm{d}a}\, a^{\alpha+3} = -\,3\, w\, a^{\alpha+2}$$

$$\alpha + 3 = -\,3\, w$$

$$\alpha = -\,3(w + 1)$$

Dust: $w = 0$. $\rho = \rho_0\, a^{-3}$. Very reasonable. The density goes down as the volume goes up.

Radiation: $w = \frac{1}{3}$. $\rho = a^{-4}$. The wavelength grows as the universe expands. Physical!