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Isometry: a coordinate transformation $x'^{\mu} = x^{\mu} + \xi^{\mu}(x)$, which we think of as infinitesimal. The term isometry applies to any transformation that leaves the metric of the same form. The metric is *form invariant* under such a transformation. We will, however, only consider continuous symmetries.

A vector $\xi^{\mu}(x)$ with these properties is called a Killing vector, and it satisfies $\mathrm{D}_{(\mu}\xi_{\nu)} = 0$. This is very restrictive. We will see that soon.

EXAMPLE: Flat, two-dimensional space. $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2$. How do we find the Killing vectors?

$$\begin{cases} x' = x + f(x, y) \\ y' = y + g(x, y) \end{cases}$$

f is ξ^x and g is ξ^y. We assume that f and g are small, so that we only need to consider them to first order. In these coordinates the affine connection vanishes, so

$$\mathrm{D}_{(\mu}\xi_{\nu)} = 0 \quad \Rightarrow \quad \begin{cases} \partial_x \xi_x = 0 \\ \partial_y \xi_y = 0 \\ \partial_x \xi_y + \partial_y \xi_x = 0 \end{cases}$$

Three equations for two unknown functions: very restrictive. Looks even over-determined. In general, in D dimensions, we get $D\,(D+1)/2$ equations for D components of the D-dimensional vector. Let us use f and g:

$$\begin{cases} \partial_x f = 0 & \Rightarrow \; f = f(y), \text{ a function of } y \text{ only.} \\ \partial_y g = 0 & \Rightarrow \; g = g(x) \\ \partial_x g + \partial_y f = 0 & \Rightarrow \; g'(x) + f'(y) = 0 \end{cases}$$

$g'(x)$ depends only on x, $f'(y)$ depends only on y, thus, given the last equation, they must both be constant. $g'(x) = -a$, $f'(y) = a$.

$$\begin{cases} f = b + a\,y \\ g = c - a\,x \end{cases}, \quad \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + a \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

b represents moving the coordinate system in the x direction. c represents translation in the y direction. a represents rotation around the origin. We have three linearly independent solutions (three parameters). $3 = D\ (D+1)/2$. The maximal number of isometries in any dimension is $D\,(D+1)/2$. This may be a simple example, but it illustrates the procedure. With a more complicated metric, this is not trivial.

EXAMPLE: $\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\varphi^2 \overset{!}{=} \mathrm{d}r'^2 + r'^2\,\mathrm{d}\varphi'^2$, with

$$\begin{cases} r' = r + f(r, \varphi) \\ \varphi' = \varphi + g(r, \varphi) \end{cases}$$

$$\begin{cases} \mathrm{d}r' = \mathrm{d}r + \partial_r f\,\mathrm{d}r + \partial_\varphi f\,\mathrm{d}\varphi \\ \mathrm{d}\varphi' = \mathrm{d}\varphi + \partial_r g\,\mathrm{d}r + \partial_\varphi g\,\mathrm{d}\varphi \end{cases}$$

Expand to linear order in the functions f and g:

$$\mathrm{d}s^2 = \mathrm{d}r^2 + 2\,\mathrm{d}r\,(\partial_r f\,\mathrm{d}r + \partial_\varphi f\,\mathrm{d}\varphi) + r^2\,\mathrm{d}\varphi^2 + 2\,r\,f\,\mathrm{d}\varphi^2 + r^{2\prime}\cdot 2\,\mathrm{d}\varphi\,(\partial_r g\,\mathrm{d}r + \partial_\varphi g\,\mathrm{d}\varphi)$$

We get one equation each for the coefficients of $\mathrm{d}r^2$, $\mathrm{d}\varphi^2$ and $\mathrm{d}r\,\mathrm{d}\varphi$. If we want to compare to the other version, $\mathrm{D}_{(\mu}\xi_{\nu)} = 0$, we have have to keep in mind that the functions here are ξ^μ not ξ_μ. We leave completing the calculation as an exercise. ♩♫

ξ_μ, $\mathrm{D}_\mu \xi_\nu$. By the Killing equation, this is antisymmetric. $\mathrm{D}_\mu \xi_\nu = \mathrm{D}_{[\mu} \xi_{\nu]}$. $\mathrm{D}_\mu \mathrm{D}_\nu\ \xi_\lambda$ will be determined in terms of $\xi_\mu, \mathrm{D}_\mu \xi_\nu$, as will any higher derivatives of a Killing vector.

For any vector we can write $0 = \mathrm{D}_{[\mu}\, \mathrm{D}_\nu\ \xi_{\lambda]}$. Anti-symmetric in the first means we get curvature

$$\mathrm{D}_{[\mu} \mathrm{D}_\nu \xi_{\lambda]} \propto R_{[\mu\nu\lambda]}{}^\sigma\ \xi_\sigma, \quad \text{but } R_{[\mu\nu\lambda]}{}^\sigma = 0$$

$$0 = 6\, \mathrm{D}_{[\mu}\, \mathrm{D}_\nu\, \xi_{\lambda]} = \mathrm{D}_\mu \mathrm{D}_\nu\, \xi_\lambda + \mathrm{D}_\nu \mathrm{D}_\lambda\, \xi_\mu + \mathrm{D}_\lambda \mathrm{D}_\mu\, \xi_\nu - \mathrm{D}_\mu \mathrm{D}_\lambda\, \xi_\nu - \mathrm{D}_\nu \mathrm{D}_\mu\, \xi_\lambda - \mathrm{D}_\lambda \mathrm{D}_\nu\, \xi_\mu =$$

Now, we use the Killing equation: $-\mathrm{D}_\lambda \xi_\nu = +\mathrm{D}_\nu \xi_\lambda$.

$$= 2\, (\mathrm{D}_\mu\, \mathrm{D}_\nu\, \xi_\lambda + \underbrace{\mathrm{D}_\nu\, \mathrm{D}_\lambda\, \xi_\mu - \mathrm{D}_\lambda\, \mathrm{D}_\nu\, \xi_\mu}_{= R_{\nu\lambda\mu}{}^\sigma \xi_\sigma})$$

$$\Rightarrow \mathrm{D}_\mu\, \mathrm{D}_\nu \xi_\lambda = -\, R_{\nu\lambda\mu}{}^\sigma\ \xi_\sigma$$

ξ_μ and $\mathrm{D}_\mu \xi_\nu$ in one point determines $\xi_\mu(x)$. If it exists.

In some point $\xi_\mu(x_0)$ and $\mathrm{D}_{[\mu} \xi_{\nu]}(x_0)$. That is all information I am allowed to put in. These are not functions, these are just numbers. Taken in one single point. $\xi_\mu(x_0)$ is D numbers. $\mathrm{D}_{[\mu} \xi_{\nu]}(x_0)$ is $D\,(D-1)/2$ numbers. In total: $D\,(D+1)/2$.

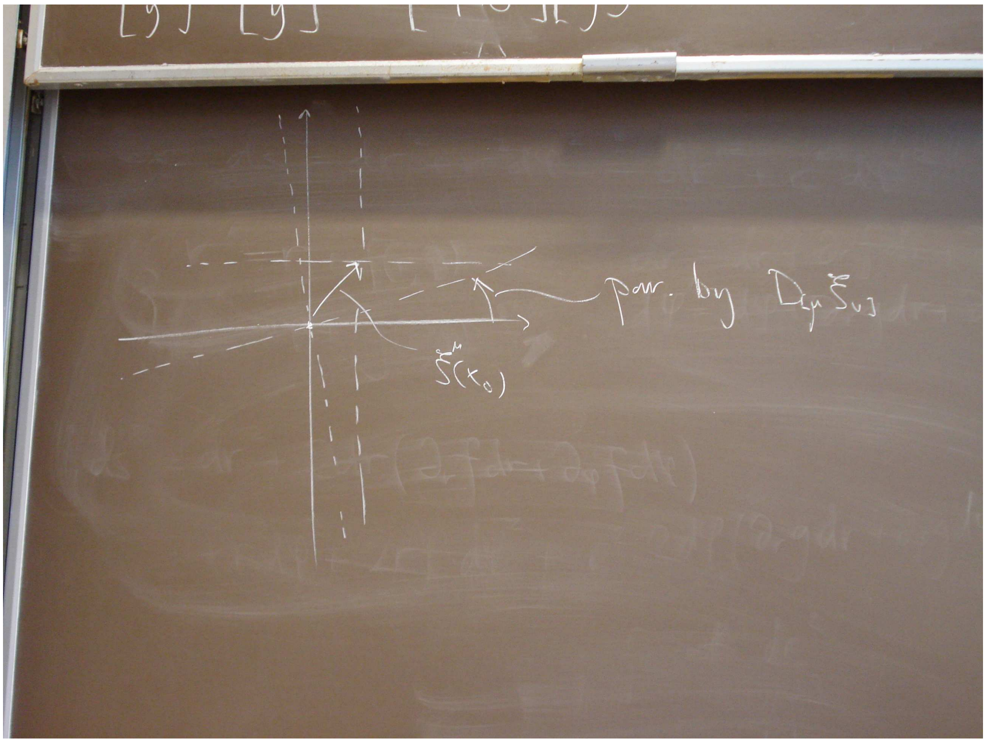

Figure 1. $\xi^\mu(x_0)$ interpreted as a translation. This is drawn in a coordinate system where the affine connection vanishes at x_0, $\Gamma^\mu_{\nu\lambda}(x_0) = 0$. In such a coordinate system $\xi^\mu(x) \approx a^\mu + \lambda^\mu{}_\nu\ (x - x_0)^\nu + \cdots$, where $\lambda^\mu{}_\nu$ is antisymmetric. The rotation angle is parametrised by $\mathrm{D}_{[\mu} \xi_{\nu]}$.

The universe: Before we try to solve Einstein's equations, we would like to have a good ansatz. From a lot of observations, it looks like the universe is

- Homogeneous — "looks essentially the same from every point in space". Translation: look at things from a different point.

- Isotropic — "looks the same in all directions". This is symmetry under rotations.

The spatial metric of the universe, at a given time t, is maximally symmetric. $D\,(D+1)/2 = 6$ Killing vectors.

$$\mathrm{D}_\mu \mathrm{D}_\nu \xi_\lambda = -R_{\nu\lambda\mu}{}^\sigma\ \xi_\sigma$$

$$\Rightarrow \quad \mathrm{D}_{[\rho}\,\mathrm{D}_{\mu]}\mathrm{D}_\nu\,\xi_\lambda = -\,\mathrm{D}_{[\rho}R_{\mu]}{}^\sigma{}_{\nu\lambda}\ \xi_\sigma - R_{\nu\lambda[\mu}{}^\sigma\ \mathrm{D}_{\rho]}\xi_\sigma$$

$$\mathrm{D}_{[\rho}\,\mathrm{D}_{\mu]}\mathrm{D}_\nu\,\xi_\lambda = \frac{1}{2}\,R_{\rho\mu\nu}{}^\sigma\ \mathrm{D}_\sigma\,\xi_\lambda + \frac{1}{2}\,R_{\rho\mu\lambda}{}^\sigma\,\mathrm{D}_\nu\,\sigma_\sigma$$

We will want to use $\mathrm{D}_{[\mu}R_{\nu\lambda]\rho}{}^\sigma \equiv 0$. We want to get rid of $\mathrm{D}_{[\rho}R_{\mu]}{}^\sigma{}_{\nu\lambda}\ \ \xi_\sigma$, so that we only have things containing $\mathrm{D}_\mu\xi_\nu$. $\mathrm{D}_{[\mu}\xi_{\nu]}$ If it is maximally symmetric, this can be chosen, for some Killing vector, to be *any* antisymmetric matrix at any given point. $\Rightarrow$ Expression with $R_{\dots} = 0$.

What we end up with (Weinberg does it in full) is the following:

$$0 = R_{\rho\mu[\nu}{}^{[\sigma}\ \delta^{\tau]}_{\lambda]} + R_{\nu\lambda[\rho}{}^{[\sigma}\ \delta^{\tau]}_{\mu]}$$

(I cannot draw this conclusion for any space that is less than maximally symmetric.) Contract this with $4\delta^\lambda_\tau$:

$$0 = D\,R_{\rho\mu\nu}{}^\sigma - R_{\rho\mu\nu}{}^\sigma - R_{\rho\mu\nu}{}^\sigma - 2\,R_{\nu[\mu\rho]}{}^\sigma - 2R_{\nu[\rho}\delta^\sigma_{\mu]}$$

where we have used $R_{[\mu\nu\lambda]}{}^\sigma = 0$ to grey out two terms.

$$(D-1)\,R_{\rho\mu\nu\sigma} = 2\,R_{\nu[\rho}g_{\mu]\sigma}$$

We can express Riemann in terms of Ricci.

$\times\, g^{\nu\rho} \Rightarrow$

$$(D-1)R_{\mu\sigma} = R\,g_{\mu\sigma} - R_{\mu\sigma} \quad \Rightarrow \quad D\,R_{\mu\sigma} = R\,g_{\mu\sigma}$$

$$R_{\mu\nu} = \frac{1}{D}\,g_{\mu\nu}\,R$$

$$R_{\mu\nu\rho\sigma} = \frac{1}{D(D-1)}\,(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}\,g_{\nu\rho})\,R$$

Remember:

$$\mathrm{D}_\nu\!\left(R^{\mu\nu} - \frac{1}{2}\,g^{\mu\nu}\,R\right) = \left(\frac{1}{D} - \frac{1}{2}\right)g^{\mu\nu}\,\mathrm{D}_\nu R = 0$$

Unless $D = 2$, we have $\mathrm{D}_\nu R = \partial_\nu R = 0$. $R = \mathrm{const} = -K$. (The minus sign: Using Weinberg's conventions, a space with positive curvature, such as a sphere, has a negative curvature scalar R.)

Given D, K and signature, a maximally symmetric space is unique. We won't prove this. The essential information of K is the sign, $+, 0, -$.

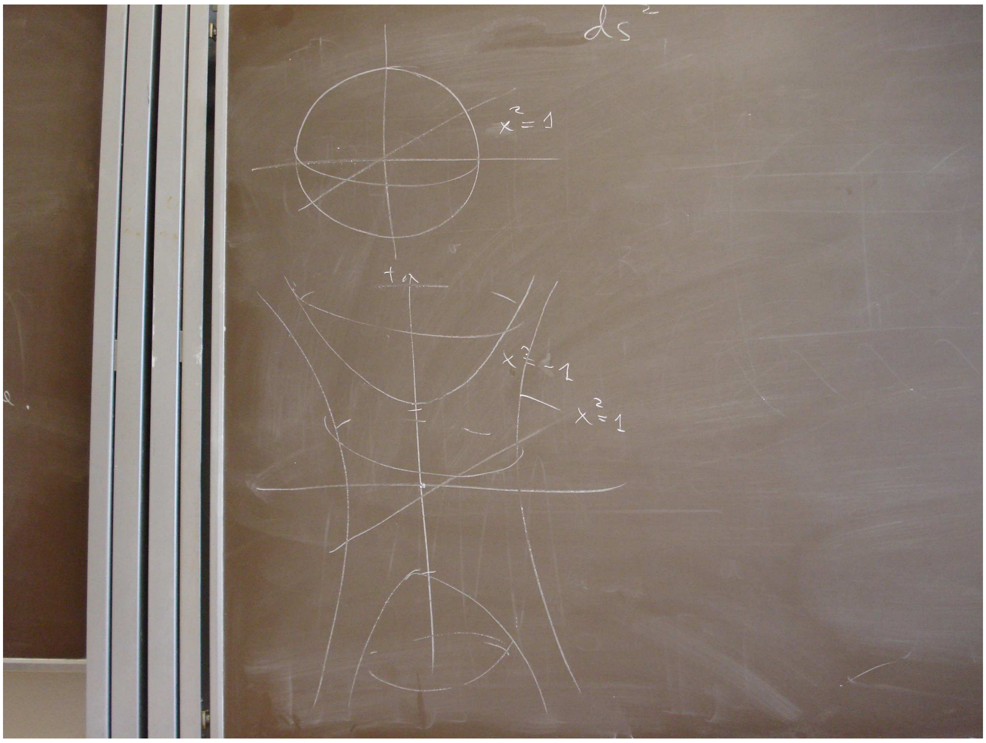

Figure 2. Maximally symmetric surfaces embedded in a) Euclidean 3D space, b) Minkowski space.

Strategy: embed D-dimensional space (-time) in a flat space (-time) of dimension $D+1$.