2008–11–26

Lecturer: Fredrik Ohlsson

The topic of today is again the Schwarzschild solution, and treat a bit more advanced topics.

6.1 Calculate the deflection of a ray of light grazing the surface of the sun.

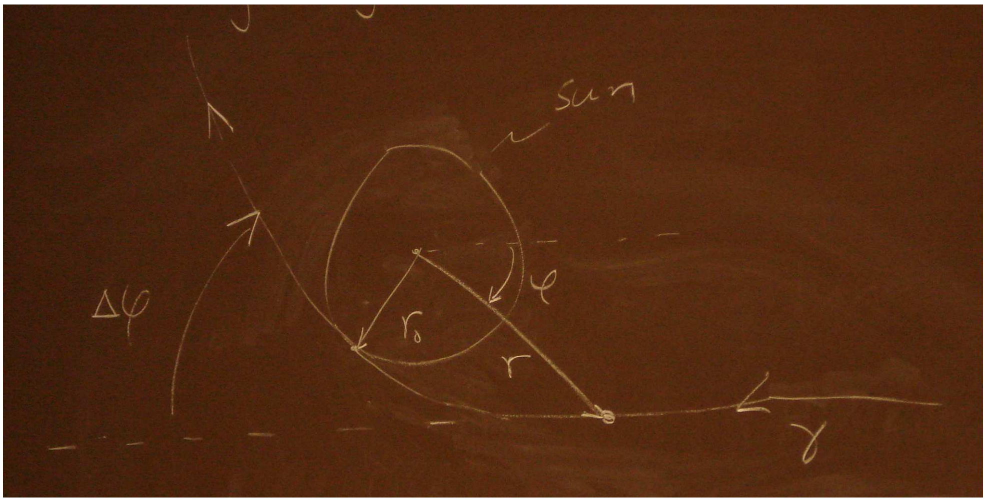

Figure 1.

$$\mathrm{d}\tau^2 = B(r)\,\mathrm{d}t^2 - A(r)\,\mathrm{d}r^2 - r^2\,\mathrm{d}\Omega$$

$$B(r) = \frac{1}{A(r)} = \left(1 - \frac{2\,MG}{r}\right)$$

Equations for photon, in the $\theta = \frac{\pi}{2}$ plane:

$$r^2\,\frac{\mathrm{d}\varphi}{\mathrm{d}t} = J\,B(r) \tag{1}$$

$$0 = \frac{1}{B(r)} - \frac{A(r)}{B(r)^2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 - \frac{J^2}{r^2} \tag{2}$$

The deflection $\Delta\varphi$ is

$$\Delta\varphi = 2\int_{r_0}^{\infty} \frac{\mathrm{d}\varphi}{\mathrm{d}r}\,\mathrm{d}r - \pi$$

The factor of 2 here is a symmetry factor. $\Delta\varphi = \Delta\tilde{\varphi} - \pi$, where $\Delta\tilde{\varphi}$ is the total change in angle. $\Delta\tilde{\varphi} = \pi$ if there is no deflection, $\Delta\varphi = 0$ if there is no deflection.

Use the product rule

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}\varphi}\,\frac{\mathrm{d}\varphi}{\mathrm{d}t}$$

in (1) and (2)

$$\Rightarrow 0 = B(r)^{-1} - A(r)\, B(r)^{-2}\, \frac{J^2\, B(r)^2}{r^4}\left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2 - \frac{J^2}{r^2}$$

$$\Rightarrow 0 = \frac{r^4}{A(r)\, B(r)\, J^2} - \left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2 - \frac{r^2}{A(r)}$$

$$\Rightarrow \left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2 = \frac{r^2}{A(r)}\left(\frac{r^2}{B(r)\, J^2} - 1\right)$$

$$\Rightarrow \left(\frac{\mathrm{d}\varphi}{\mathrm{d}r}\right) = \pm\, \frac{1}{r}\, \frac{\sqrt{A(r)}}{\sqrt{\frac{r^2}{B(r)\, J^2} - 1}}, \qquad \begin{array}{l} +:\ \mathrm{d}r > 0 \ \Rightarrow\ \mathrm{d}\varphi > 0 \ \text{ after grazing} \\ -:\ \mathrm{d}r > 0 \ \Rightarrow\ \mathrm{d}\varphi < 0 \ \text{ before grazing} \end{array}$$

We are interested in the positive version. Determine J using $\mathrm{d}r/\mathrm{d}\varphi = 0$ at $r = r_0$.

$$\Rightarrow \frac{r_0^2}{B(r_0)\, J^2} - 1 = 0 \quad \Rightarrow \quad J^2 = \frac{r_0^2}{B(r_0)}$$

$$\Rightarrow \quad \Delta\tilde{\varphi} = 2 \int_{r_0}^{\infty} \mathrm{d}r\, \frac{\sqrt{A(r)}}{r\sqrt{\frac{r^2}{r_0^2}\frac{B(r_0)}{B(r)} - 1}}$$

In general this can only be solved numerically. In the limit $r_0 \gg 2\,MG \quad (\Rightarrow r \gg 2\,MG)$.

$$\sqrt{A(r)} = \frac{1}{\sqrt{1 - \frac{2\,MG}{r}}} \simeq 1 + \frac{MG}{r}$$

$$\frac{r^2}{r_0^2}\frac{B(r_0)}{B(r)} - 1 = \frac{r^2}{r_0^2}\left(1 - \frac{2\,MG}{r_0}\right)\left(1 + \frac{2\,MG}{r}\right) - 1 \simeq \frac{r^2}{r_0^2}\left(1 - \frac{2\,MG}{r_0} + \frac{2\,MG}{r}\right) - 1 =$$

$$= \left(\frac{r^2}{r_0^2} - 1\right) - \frac{2MG\,r^2}{r_0}\left(\frac{1}{r_0} - \frac{1}{r}\right) = \ldots = \frac{1}{r_0^2}(r^2 - r_0^2)\left(1 - \frac{2\,MG\,r}{r_0(r_0 + r)}\right)$$

$$\Rightarrow \Delta\tilde{\varphi} \simeq 2 \int_{r_0}^{\infty} \mathrm{d}r\, \frac{1}{r}\left(1 + \frac{MG}{r}\right)\frac{r_0}{\sqrt{r^2 - r_0^2}}\left(1 + \frac{MG\,r}{r_0(r + r_0)}\right) \simeq$$

$$\simeq 2\, r_0 \int_{r_0}^{\infty} \mathrm{d}r\, \frac{1}{r\sqrt{r^2 - r_0^2}}\left(1 + \frac{MG}{r} + \frac{MG\,r}{r_0(r + r_0)}\right) = \left[\begin{array}{c}\text{table of}\\ \text{integrals}\end{array}\right] =$$

$$= 2\, r_0\left[\frac{1}{r_0}\arccos\left(\frac{r_0}{r}\right) + \frac{MG\sqrt{r^2 - r_0^2}}{r\, r_0^2} + \frac{MG}{r_0^2}\frac{\sqrt{r - r_0}}{\sqrt{r + r_0}}\right]_{r_0}^{\infty}$$

$$= 2\, r_0\left(\frac{1}{r_0}\left(\frac{\pi}{2} - 0\right) + \frac{MG}{r_0}(1 - 0) + \frac{MG}{r_0^2}(1 - 0)\right) = \pi + 4\,\frac{MG}{r_0^2}$$

$$\Delta\varphi = \Delta\tilde{\varphi} - \pi \simeq \frac{4MG}{r_0}$$

To get a numerical value we must reinsert c.

$$[M] = \mathrm{kg}, \quad [r_0] = m, \quad [G] = \frac{\mathrm{kg\,m}}{\mathrm{s}^2}\frac{\mathrm{m}^2}{\mathrm{kg}^2} = \frac{\mathrm{m}^3}{\mathrm{kg\,s}^2}$$

$$\Rightarrow \left[\frac{4MG}{r_0}\right] = \frac{\mathrm{m}^2}{\mathrm{s}^2} \Rightarrow \text{ a factor of } c^2$$

$$\Rightarrow \Delta\varphi = \frac{4\,MG}{r_0\,c^2}$$

For the sun this gives $\Delta\varphi = 1,75'' = 1,75°/3600$.

6.2

> "Find the time for one revolution in the circular orbit at radius $r_c = 3\,a = 6\,MG$ as measured by
>
> (a) a comoving observer
>
> (b) an observer at rest at $r = r_c$
>
> (c) a distant ($r \to \infty$) observer at rest."

a) Comoving observer. Recall 5.3: $r = r_c = 6\,MG \Rightarrow j^2 = 12\,M^2\,G^2$.

$$\theta = \frac{\pi}{2}, \quad \dot{\theta} = 0, \quad \dot{r} = 0, \quad \dot{\varphi} = \frac{j^2}{r_c^2} = \frac{\sqrt{12}\,MG}{36\,M^2\,G^2} = \frac{1}{6\sqrt{3}\,MG}$$

Lagrangian (massive particle):

$$L = 1 = \left(1 - \frac{2\,MG}{r_c}\right)\dot{t}^2 - \cancel{(..)\dot{r}^2} - \cancel{r_c^2\dot{\theta}^2} - r_c^2\underbrace{\sin^2\theta}_{=1}\,\dot{\varphi}^2 =$$

$$= \left(1 - \frac{2\,MG}{6\,MG}\right)\dot{t}^2 - 36\,M^2\,G^2 \cdot \frac{1}{3\cdot 36\,M^2\,G^2} = \frac{2}{3}\dot{t}^2 - \frac{1}{3}$$

$$\Rightarrow \dot{t}^2 = 2 \quad \Rightarrow \quad t = \sqrt{2}\,\tau$$

Proper time measured by the comoving observer: $\tau_c = t_0/\sqrt{2}$, where t_0 is coordinate time. To get τ_c use

$$\dot{\varphi} = \frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = \frac{1}{6\sqrt{3}\,MG}$$

$$\Rightarrow \quad \tau_c = 6\sqrt{3}\,MG\int_0^{2\pi}\mathrm{d}\varphi = 12\sqrt{3}\,\pi\,MG$$

b) Observer at rest at $r = r_c$.

$$\theta = \frac{\pi}{2}, \quad \dot{\theta} = 0, \quad \dot{r} = 0, \quad \dot{\varphi} = 0$$

$$\Rightarrow L = 1 = \left(1 - \frac{2\,MG}{r_c}\right)\dot{t}^2 \quad \Rightarrow 1 = \frac{2}{3}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2$$

so the proper time measured by an observer at rest is

$$\Rightarrow \tau_r = \sqrt{\frac{2}{3}}\,t_0 = \sqrt{\frac{2}{3}}\,12\sqrt{6}\,\pi\,MG = 24\,\pi\,MG > \tau_c$$

c) Observer at rest at $r \to \infty$.

$$\dot{\theta} = \dot{r} = \dot{\varphi} = 0$$

$$\Rightarrow L = 1 = \left(1 - \frac{2\,MG}{r}\right)\dot{t}^2 \longrightarrow \dot{t}^2 \text{ as } r \to \infty.$$

$$\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 = 1$$

The proper time measured by an observer at $r \to \infty$ is just the coordinate time:

$$\Rightarrow \tau_\infty = t_0 = 12\sqrt{6}\,\pi\,MG > \tau_\mathrm{c}$$

Comparing all these proper times:

$$\Rightarrow \quad \tau_\mathrm{c} < \tau_\mathrm{r} < \tau_\infty = t_0$$

6.3

> "Calculate the coordinate time needed for a particle to fall into a black hole and pass the event horizon. Compare with the proper time measured by a comoving observer."

Equations ($\theta = \pi/2, \dot{\theta} = 0, \dot{\varphi} = 0$: radial motion)

$$\begin{cases} E = \frac{1}{B(r)} - \frac{A(r)}{B(r)^2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 - \cancel{\frac{J^2}{r^2}} \\ \mathrm{d}\tau^2 = E\,B(r)^2\,\mathrm{d}t^2 \end{cases}$$

$$\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 = \frac{B(r)^2}{A(r)\,B(r)} - E\,\frac{B(r)}{A(r)} = B(r)^2(1 - E\,B(r))$$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -\,B(r)\sqrt{1 - E\,B(r)}$$

Negative, since we are falling into the black hole.

Now consider particles close to the event horizon.

$$r = r_\mathrm{S} + x = 2\,MG + x$$

$x \ll 2\,MG$. $x = 0$ at the event horizon.

$$\Rightarrow B(r) = \left(1 - \frac{2\,MG}{r}\right) = 1 - \frac{2\,MG}{2\,MG + x} \simeq \frac{x}{2\,MG}$$

$$\Rightarrow \frac{\mathrm{d}r}{\mathrm{d}t} = -\,\frac{x}{2\,MG}\sqrt{1 - \frac{E\,x}{2\,MG}} \simeq -\,\frac{x}{2\,MG}\left(1 - \frac{E_x}{2\,MG}\right) \simeq -\,\frac{x}{2\,MG} = \frac{\mathrm{d}x}{\mathrm{d}t}$$

$$\dot{x} + \frac{x}{2\,MG} = 0$$

$$x(t) = R_0\,\mathrm{e}^{-t/2MG}$$

It will take an infinite amount of coordinate time for the particle to reach the event horizon. An observer at infinity will *never* see the particle enter the black hole. (It will just fade away.)

The comoving observer:

$$\mathrm{d}\tau^2 = E\,B(r)^2\,\mathrm{d}t^2$$

$$\tau = \sqrt{E}\int_{t_0}^{t_1} B(r)\,\mathrm{d}t = \sqrt{E}\int_{r_0}^{r_1} B(r)\,\frac{\mathrm{d}t}{\mathrm{d}r}\,\mathrm{d}r = \left[\frac{\mathrm{d}t}{\mathrm{d}r} = -\frac{1}{B(r)\,\sqrt{1-E\,B(r)}}\right] =$$

$$= -\sqrt{E}\int_{R_0}^{2MG} \frac{\mathrm{d}r}{\sqrt{1-B(r)\,E}} = -\sqrt{E}\int_{R_0}^{2MG} \frac{\mathrm{d}r}{\sqrt{1-E\left(1-\frac{2\,M\,G}{r}\right)}}$$

This is a finite number. In the interval $[R_0, 2\,M\,G]$ r is close to $r_{\mathrm{S}} = 2\,M\,G \Rightarrow R_0 - 2\,M\,G \ll 1$.

$$\tau \simeq -\sqrt{E}\int_{R_0}^{2MG} \mathrm{d}r = \sqrt{E}(R_0 - 2\,M\,G) < \infty$$

This means that the particle reaches (and passes) the event horizon in finite proper time.