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Gravitational radiation

Just like in electromagnetism, where we considered Maxwell's equations, the Einstein equations have radiative solutions — a fluctuation in the curvature of space-time (that is, a fluctuation in the metric $g_{\mu\nu}$) that propagates as a wave.

Is this something we are likely to observe, when we look out into the universe? Perhaps not. These waves would be extremely weak, compared to everything else we consider in physics. There are some processes, like atomic processes, that *could* occur through gravitational interactions, but the probability would be on the order of 10^{-54} per second — not comparable in any way to the normal electromagnetic interactions. But one might imagine heavy stuff in space that could give measurable gravitational radiation. Perhaps a binary system of black holes, orbiting each other and changing the metric around them as they go.

More fundamentally, radiative solutions are essential for the concept of elementary particles — it is needed for the *graviton*. This particle mediates the gravitational force. Whenever you see the graviton, think of the photon — that is the analogy with electromagnetism. But there are important differences compared to electromagnetism: The gravitational field is self-interacting: $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu}$. Gravity will act as its own source. This is not the case in electromagnetism. In electromagnetism the electric charge is the source, but the photon has no electric charge and, thus, does not interact with other photons.

The weak field approximation

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

The metric is not quite the Minkowski metric $\eta_{\mu\nu}$ — but close! $|h| \ll 1$. We call h a fluctuation. We consider the perturbation small enough for us to use the ordinary Minkowski metric to raise and lower indices. Expand to linear order in h:

$$g^{\mu\nu} \simeq \eta^{\mu\nu} - h^{\mu\nu}$$

$$g^{\mu\nu} g_{\nu\rho} \simeq \ldots \simeq \delta^\mu_\rho + \mathcal{O}(h^2)$$

Consider the quantities that we need in the field equations. The first thing we need is the affine connection:

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2}\, g^{\mu\eta} (\partial_\nu g_{\eta\rho} + \partial_\rho g_{\eta\nu} - \partial_\eta g_{\nu\rho})$$

$\partial_\nu g_{\mu\rho}$ is already linear in h.

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2}\, \eta^{\mu\eta} (\partial_\nu h_{\eta\rho} + \partial_\rho h_{\eta\nu} - \partial_\eta h_{\nu\rho}) + \mathcal{O}(h^2) =$$

$$= \frac{1}{2} \big(\partial_\nu h^\mu{}_\rho + \partial_\rho h^\mu{}_\nu - \partial^\mu h_{\nu\rho} \big)$$

$$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu} = \partial_\nu \Gamma^\lambda_{\lambda\mu} - \partial_\lambda \Gamma^\lambda_{\mu\nu} + \underbrace{\Gamma\,\Gamma - \Gamma\,\Gamma}_{=\mathcal{O}(h^2)} \simeq$$

$$\simeq \partial_\nu \left(\frac{1}{2} \big(\cancel{\partial_\lambda h^\lambda{}_\mu} + \partial_\mu h^\lambda{}_\lambda - \cancel{\partial^\lambda h_{\mu\nu}} \big) \right) - \partial_\lambda \left(\frac{1}{2} \big(\partial_\mu h^\lambda{}_\nu + \partial_\nu h^\lambda{}_\mu - \partial^\lambda h_{\mu\nu} \big) \right) =$$

$\Box = \partial_\mu \partial^\mu$,

$$= \frac{1}{2} \left(\Box h_{\mu\nu} + \partial_\mu \partial_\nu h^\lambda{}_\lambda - \partial_\lambda \partial_\mu h^\lambda{}_\nu - \partial_\lambda \partial_\nu h^\lambda{}_\mu \right)$$

Einstein equation on the form

$$R_{\mu\nu} = -\,8\,\pi\,G\underbrace{\left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T^{\lambda}{}_{\lambda}\right)}_{\equiv S_{\mu\nu}}$$

$S_{\mu\nu}$ is the source of the gravitational radiation. The linearised form of Einstein's equations is

$$\Box h_{\mu\nu} + \partial_\mu\partial_\nu h^\lambda{}_\lambda - \partial_\lambda\partial_\mu h^\lambda{}_\nu - \partial_\lambda\partial_\nu h^\lambda{}_\mu = -\,16\pi\mathrm{G}\mathrm{S}_{\mu\nu} \tag{1}$$

A solution to this equation is not unique. We have gauge invariance: $x^\mu \to x^\mu + \varepsilon^\mu(x)$, $\partial_\mu\varepsilon(x) \sim \mathcal{O}(h)$. $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \to g'_{\mu\nu} = \eta_{\mu\nu} + h'_{\mu\nu} =$

$$= \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\,g_{\rho\sigma} = \left(\delta^\rho_\mu - \partial_\mu\varepsilon^\rho\right)\left(\delta^\sigma_\nu - \partial_\nu\varepsilon^\sigma\right)\left(\eta_{\rho\sigma} + h_{\rho\sigma}\right) =$$

$$= \eta_{\mu\nu} + \underbrace{h_{\mu\nu} - \partial_\mu\varepsilon_\nu - \partial_\nu\varepsilon_\mu}_{= h'_{\mu\nu}} + \mathcal{O}(h^2)$$

If $h_{\mu\nu}$ is a solution to (1), then so is $h'_{\mu\nu}$.

Now comes the first comparison to electromagnetism. This is exactly what happens in electromagnetism. What we have in electromagnetism is the vector potential A_μ. If we make a transformation of A_μ we get $A'_\mu = A_\mu + \partial_\mu\Lambda$ for some scalar Λ. This means that the field tensor $F^{\mu\nu} \to F'^{\mu\nu} = F^{\mu\nu}$. The field tensor is invariant, and thus the field equations are invariant. This is the same thing here. Say that $h_{\mu\nu}$ solves the equation. Then $h'_{\mu\nu}$ as defined above, also solve the field equation. We can get an infinite number of solutions, if we have one.

But we don't have any solution. To get rid of the redundancy we *fix the gauge*. That means that we fix a coordinate system. We will use the *harmonic gauge*. The condition here is that

$$g^{\mu\nu}\,\Gamma^\lambda_{\mu\nu} = 0$$

How does this carry to the weak field approximation?

$$g^{\mu\nu}\Gamma^\lambda_{\mu\nu} = \frac{1}{2}\,\eta^{\mu\nu}\left(\partial_\mu h^\lambda{}_\nu + \partial_\nu h^\lambda{}_\mu - \partial^\lambda\, h_{\mu\nu}\right) + \mathcal{O}(h^2) = 0$$

$$\Rightarrow \quad \boxed{\partial_\mu h^\mu{}_\lambda = \frac{1}{2}\,\partial_\lambda h^\mu{}_\mu}$$

This simplifies things enormously in this case. If we compare this to the left hand side of (1), we see that "all the nonsense here" vanishes. We get $\Box h_{\mu\nu}$ equals a source term, which looks a bit less scary. The linearised Einstein's equations in harmonic gauge:

$$\begin{cases} \Box h_{\mu\nu} = -\,16\pi\,G\,S_{\mu\nu} \\ \partial_\mu\,h^\mu{}_\lambda = \frac{1}{2}\,\partial_\lambda h^\mu{}_\mu \end{cases}$$

An inhomogeneous partial differential equation. We need one particular solution to the full equation $\Box h_{\mu\nu} = -\,16\pi\,G\,S_{\mu\nu}$, and then the solutions to the corresponding homogeneous equation.

1. The particular solution is found using a Green's function:

$$h^{\mathrm{ret}}_{\mu\nu}(x) = 4G\int \mathrm{d}^3\boldsymbol{x}'\,\frac{S_{\mu\nu}(\boldsymbol{x}', t - |\boldsymbol{x} - \boldsymbol{x}'|)}{|\boldsymbol{x} - \boldsymbol{x}'|}$$

The gauge condition is satisfied automatically by this solution, since $\partial_\mu T^{\mu\nu} = 0$.

2. The plane wave solution:

$$\begin{cases} \Box h_{\mu\nu} = 0 \\ \partial_\mu h^\mu{}_\lambda = \frac{1}{2}\,\partial_\lambda h^\mu{}_\mu \end{cases}$$

General form $h_{\mu\nu}(x) = \varepsilon_{\mu\nu}(x)\,\mathrm{e}^{\mathrm{i}k\cdot x}$ + complex conjugate, where $\varepsilon_{\mu\nu}(x)$ is called the polarisation tensor, satisfying $\varepsilon_{\mu\nu} = \varepsilon_{\nu\mu}$. With scalar product we mean $k \cdot x = k_\mu\, x^\mu$. k_μ = momentum carried by the wave (wave number).

$$\Box h_{\mu\nu} = 0 \quad \Rightarrow \quad k^2 = k_\mu k^\mu = 0$$

This means that the wave moves with the speed of light (unit velocity). That means that the graviton is massless. Then we had the gauge condition, too:

$$\partial_\mu h^\mu{}_\lambda = \frac{1}{2}\,\partial_\lambda\, h^\mu{}_\mu \quad \Rightarrow \quad k_\mu \varepsilon^\mu{}_\lambda = \frac{1}{2}\,k_\lambda\,\varepsilon^\mu{}_\mu$$

The harmonic gauge might not describe a unique coordinate system, but restrict us to a subset of coordinate systems. There are still coordinate transformations that we can do; there is still gauge invariance. We can make a transformation $x^\mu \to x'^\mu = x^\mu + \mathrm{i}\,\xi^\mu\,\mathrm{e}^{\mathrm{i}k\cdot x}$ + complex conjugate. Without leaving harmonic gauge $\varepsilon_{\mu\nu} \to \varepsilon'_{\mu\nu} = \varepsilon_{\mu\nu} + k_\mu \xi_\nu + k_\nu \xi_\mu$.

$$\Rightarrow k_\mu \varepsilon'^\mu{}_\lambda - \frac{1}{2} k_\nu \varepsilon'^\mu{}_\mu = \varepsilon\text{-}\not{p}\text{art} + k_\mu(k^\mu \xi_\nu + k_\nu \xi^\mu) - \frac{1}{2}\,k_\nu(2\,k\cdot\xi) = k_\mu k^\mu\,\xi_\nu = 0$$

Thus, we remain in harmonic gauge.

Compare to electromagnetism: $A_\mu = \alpha_\mu \cdot \mathrm{e}^{\mathrm{i}k\cdot x}$ + complex conjugate (plane waves).

$$\alpha_\mu = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \quad \text{4 independent components.}$$

$\Box A_\mu = 0 \Rightarrow k^2 = 0$. Maxwell's equations in Lorentz gauge: $\partial_\mu A^\mu = 0 \Rightarrow k_\mu \alpha^\mu = 0$. We reduce the number of independent components, since we impose a condition on them. There is a remaining gauge symmetry: $A_\mu \to A_\mu - \partial_\lambda \lambda$ where $\lambda = \eta\ \mathrm{e}^{\mathrm{i}k\cdot x}$ + complex conjugate. Fixing this gauge removes one independent component. Thus there are only two physical modes.

$$\alpha_\mu^{\mathrm{phys}} = \begin{pmatrix} 0 \\ \alpha_1 \\ \alpha_2 \\ 0 \end{pmatrix}$$

for some choice of coordinate system. (This is propagation in the z direction.)

Returning to general relativity:

$$h_{\mu\nu}(x) = \varepsilon_{\mu\nu}(x)\,\mathrm{e}^{\mathrm{i}k\cdot x} + \text{complex conjugate}$$

$$\varepsilon_{\mu\nu} = \begin{pmatrix} \varepsilon_{00} & \cdots & \varepsilon_{03} \\ \vdots & \ddots & \vdots \\ \varepsilon_{30} & \cdots & \varepsilon_{33} \end{pmatrix}$$

$\varepsilon_{\mu\nu} = \varepsilon_{\nu\mu} \Rightarrow 10$ independent components. Einstein's equations in harmonic gauge:

$$\Box h_{\mu\nu} = 0 \quad \Rightarrow \quad k^2 = 0, \quad k^\mu = (k, 0, 0, k) \text{ for propagation in the } z \text{ direction}$$

$$k_\mu \varepsilon^\mu{}_\lambda = \frac{1}{2} k_\lambda \varepsilon^\mu{}_\mu \quad \text{(gauge)}$$

These are four independent equations relations components of k and ε with each other. This reduces the number of independent components in ε by four. Then we also had the remaining symmetry: $x^\mu \to x^\mu + \mathrm{i}\, \xi^\mu\, \mathrm{e}^{\mathrm{i}k\cdot x} + \text{complex conjugate}$, which is also four equations, giving an additional reduction in the number of components by four. This means that there are only two independent components in $\varepsilon_{\mu\nu}$:

$$\varepsilon^{\text{phys}}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \varepsilon_{11} & \varepsilon_{12} & 0 \\ 0 & \varepsilon_{12} & -\varepsilon_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Imagine a circle of massive points in a plane. Let a gravitational wave enter in a direction perpendicular to the circle. We would see some squeezing (see figure 1). It would oscillate between squeezing and stretching.

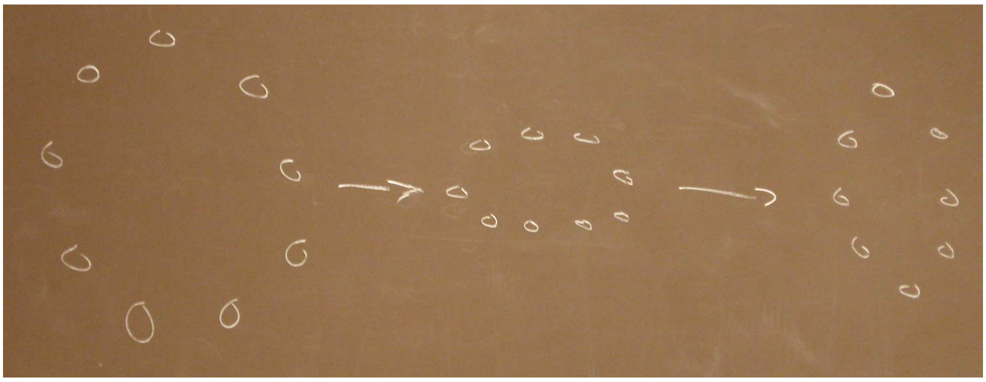

Figure 1.

Lastly, a few words on spin. Weinberg has an extensive treatment of the different components of this matrix. Group the components $\varepsilon_{\mu\nu}$ according to how they transform under rotation as follows:

$$e_\pm = \varepsilon_{11} \mp \mathrm{i}\varepsilon_{12}, \quad f_\pm = \varepsilon_{31} \mp \mathrm{i}\varepsilon_{32}, \quad \varepsilon_{00}, \quad \varepsilon_{33}$$

Consider rotation by some angle θ around the z axis (the direction of propagation):

$$e'_\pm = \exp(\mathrm{i}\, 2\theta)\, e_\pm, \quad f'_\pm = \exp(\mathrm{i}\theta)\, f_\pm, \quad \varepsilon'_{00} = \varepsilon_{00}, \quad \varepsilon'_{33} = \varepsilon_{33}$$

Helicity: If $\psi' = \exp(\mathrm{i}\, h\, \theta) \psi$, then ψ has helicity h.

$\Rightarrow$ The physical components of $\varepsilon_{\mu\nu}$ have helicity 2. Helicity can be seen as the projection of the spin on the direction of propagation. The graviton must have spin 2.