2008–11–24

Schwarzschild solution:

$$\mathrm{d}s^2 = -\,B(r)\,\mathrm{d}t^2 + A(r)\,\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2$$

where $\mathrm{d}\Omega^2 = \tilde{g}_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$ is the metric of the two-sphere. $i = 1, 2$.

$$\mathrm{d}s^2 = -\left(1 - \frac{2\,M\,G}{r}\right)\mathrm{d}t^2 + \left(\frac{1}{1 - \frac{2\,M\,G}{r}}\right)\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2$$

The affine connection is:

$$\Gamma^r_{rr} = \frac{A'}{2\,A},\quad \Gamma^r_{tt} = \frac{B'}{2\,A},\quad \Gamma^r_{ij} = -\frac{r}{A}\tilde{g}_{ij},\quad \Gamma^t_{rt} = \frac{B'}{2\,B},\quad \Gamma^i_{rj} = \frac{1}{r}\,\delta^i_j,\quad \Gamma^i_{jk} = \tilde{\Gamma}^i_{jk}$$

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}p^2} + \Gamma^\mu_{\nu\lambda}\,\frac{\mathrm{d}x^\nu}{\mathrm{d}p}\,\frac{\mathrm{d}x^\lambda}{\mathrm{d}p}$$

p is some parameter.

$$\frac{\mathrm{d}}{\mathrm{d}p}(B\,\dot{t}) \quad \rightarrow \quad \text{choose } B\,\dot{t} = 1, \quad \dot{t} = \frac{1}{B}$$

Restrict to some plane. The initial position and velocity define a plane, and just as in the Newtonian case, there is nothing that would take us out of this plane. We choose our coordinates such that $\theta = \pi/2$. For the angular motion:

$$\frac{\mathrm{d}}{\mathrm{d}p}(r^2\dot{\varphi}) = 0$$

This looks a lot like the angular momentum, classically: $m\,r^2\,\dot{\varphi}$. But note which variable we differentiate with respect to.

$$\dot{\varphi} = \frac{J}{r^2}$$

with some constant J.

$$\frac{\mathrm{d}}{\mathrm{d}p}\underbrace{\left(A\,\dot{r}^2 + \frac{J^2}{r^2} - \frac{1}{B}\right)}_{=-E} = 0$$

Proper time:

$$\mathrm{d}\tau^2 = -\,\mathrm{d}s^2$$

$$\dot{\tau}^2 = B\,\dot{t}^2 - A\,\dot{r}^2 - r^2\,\dot{\varphi}^2 = E$$

$\mathrm{d}\tau = \sqrt{E}\ \mathrm{d}p$. τ is meaningful as a time, proper time, only when the trajectory is time-like. For something that is massless, $\mathrm{d}\tau = 0$, and we cannot use τ as a parameter.

Radial motion $(\varphi = \text{constant} \Rightarrow J = 0)$

This is essentially a one-dimensional problem.

$$A\,\dot{r}^2 = \frac{1}{B} - E$$

Divide by A, and use $A\,B = 1$,

$$\dot{r}^2 = 1 - E\,B(r)$$

In terms of t:

$$\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 = B^2(r)\,(1 - E\,B(r))$$

We will look at what happens close to the horizon. That's where the interesting things happen, where things behave very non-Newtonian. Close to the horizon $r = 2\,M\,G + x$, with $x \ll 2\,M\,G$. ($x = 0$ is the horizon.)

$$B = 1 - \frac{2\,M\,G}{r} = 1 - \frac{2\,M\,G}{2\,M\,G + x} = 1 - \frac{1}{1 + \frac{x}{2\,M\,G}} \simeq \frac{x}{2\,M\,G}$$

In the $B^2(r)(1 - E\,B(r))$ we can drop the $E\,B(r)$ to lowest order in x.

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 \approx \left(\frac{x}{2\,M\,G}\right)^2$$

Choose inward motion:

$$\frac{\mathrm{d}x}{\mathrm{d}t} \approx -\frac{x}{2\,M\,G}$$

A linear equation, which is not surprising, since we have been linearising things:

$$x(t) \approx C\,\mathrm{e}^{-t/2MG}$$

This is a funny solution. As $t \to \infty$, the particle stops at $x = 0$, at the horizon. This holds when we look at the situation using the coordinate time t. But the coordinate time t behaves strangely near the horizon. Before we decide that the particle actually stops, we have to see if this is an effect of the choice of coordinates, or if it is something deeper. The time dilation relative to an observer at infinity becomes infinite. We know that inside the horizon, it is r that takes on the role of a time coordinate.

What happens if we use proper time instead?

$$\left(\frac{\mathrm{d}r}{\mathrm{d}p}\right)^2 = E\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 = 1 - E\,B(r) \simeq 1 \text{ for } x \ll 2\,M\,G$$

$\mathrm{d}r/\mathrm{d}\tau$ is some finite value. It does not go to zero as $r \to 2\ M\ G$, as it did when we used the coordinate time t.

You can't detect the horizon locally. It is a global property. But the centre is a singularity. There is some theorem that says, essentially, that a horizon implies a singularity inside.

General motion

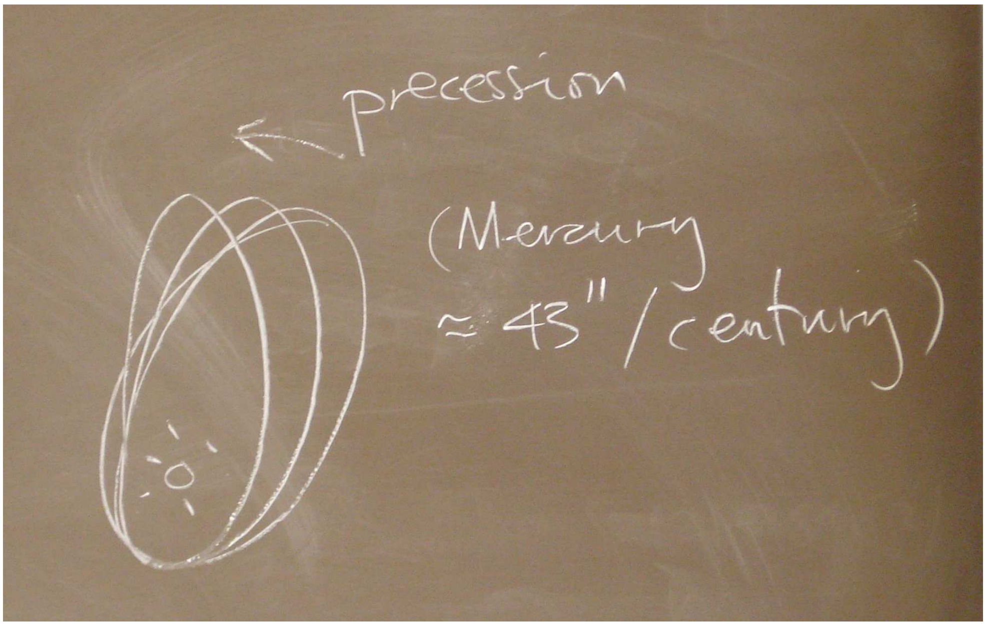

Figure 1. Precession of the orbit. For Mercury $\approx 43''$/century. [Actually, the precession is much larger than this, see e.g. Weinberg (1972). The planetary orbit is only closed in Newtonian mechanics if we neglect the influence of the other planets. However, the influence of the other planets is more significant than the corrections due to general relativity. The $43''$/century is this correction due to general relativity; it is the part of the precession that cannot be explained using Newtonian celestial mechanics.]

$$A\,\dot{r}^2 + \frac{J^2}{r^2} - \frac{1}{B} = -E$$

Use

$$\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}\varphi}\,\dot{\varphi} = \frac{\mathrm{d}r}{\mathrm{d}\varphi}\,\frac{J}{r^2}.$$

$$\frac{A}{r^4}\left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2 + \frac{1}{r^2} - \frac{1}{J^2\,B(r)} = -\frac{E}{J^2}$$

Solve for $\mathrm{d}\varphi/\mathrm{d}r$:

$$\frac{\mathrm{d}\varphi}{\mathrm{d}r} = \frac{\sqrt{A}}{r^2\sqrt{\frac{1}{J^2\,B(r)} - \frac{E}{J^2} - \frac{1}{r^2}}}$$

This leads to some kind of elliptical integral. This won't be periodic.

Look for r_- and r_+.

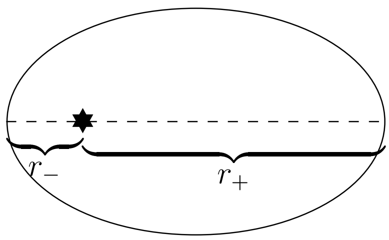

Figure 2. r_- is the perihelion (the closest approach to the sun) and r_+ is the aphelion (farthest approach to the sun). [If the central body is not the sun, the more general terms periapsis and apoapsis may be used. If the central body is the Earth, the words perigee and apogee are often used.]

$$\frac{1}{r_\pm^2} - \frac{1}{J^2 B(r_\pm)} = -\frac{E}{J^2}$$

We can use $r_\pm$ instead of E and J.

$$E = \frac{\frac{r_+^2}{B(r_+)} - \frac{r_-^2}{B(r_-)}}{r_+^2 - r_-^2}$$

$$J = \frac{\frac{1}{B(r_+)} - \frac{1}{B(r_-)}}{\frac{1}{r_+^2} - \frac{1}{r_-^2}}$$

Integrate, from r_- to some other r:

$$\varphi(r) - \varphi(r_-) = \int_{r_-}^{r} \frac{A^{1/2}}{r^2}\left(\frac{r_-^2\big(B^{-1}(r) - B^{-1}(r_-)\big) - r_+^2\big(B^{-1}(r) - B^{-1}(R_+)\big)}{r_+^2 r_-^2 (B^{-1}(r_+) - B^{-1}(r_-))} - \frac{1}{r^2}\right)^{-1/2} \mathrm{d}r$$

"It is not too bad."

Use $r_\pm / 2\,MG \gg 1$.

Calling the precession angle $\Delta\varphi$:

$$\Delta\varphi = 2\,(\varphi(r_+) - \varphi(r_-) - \pi).$$

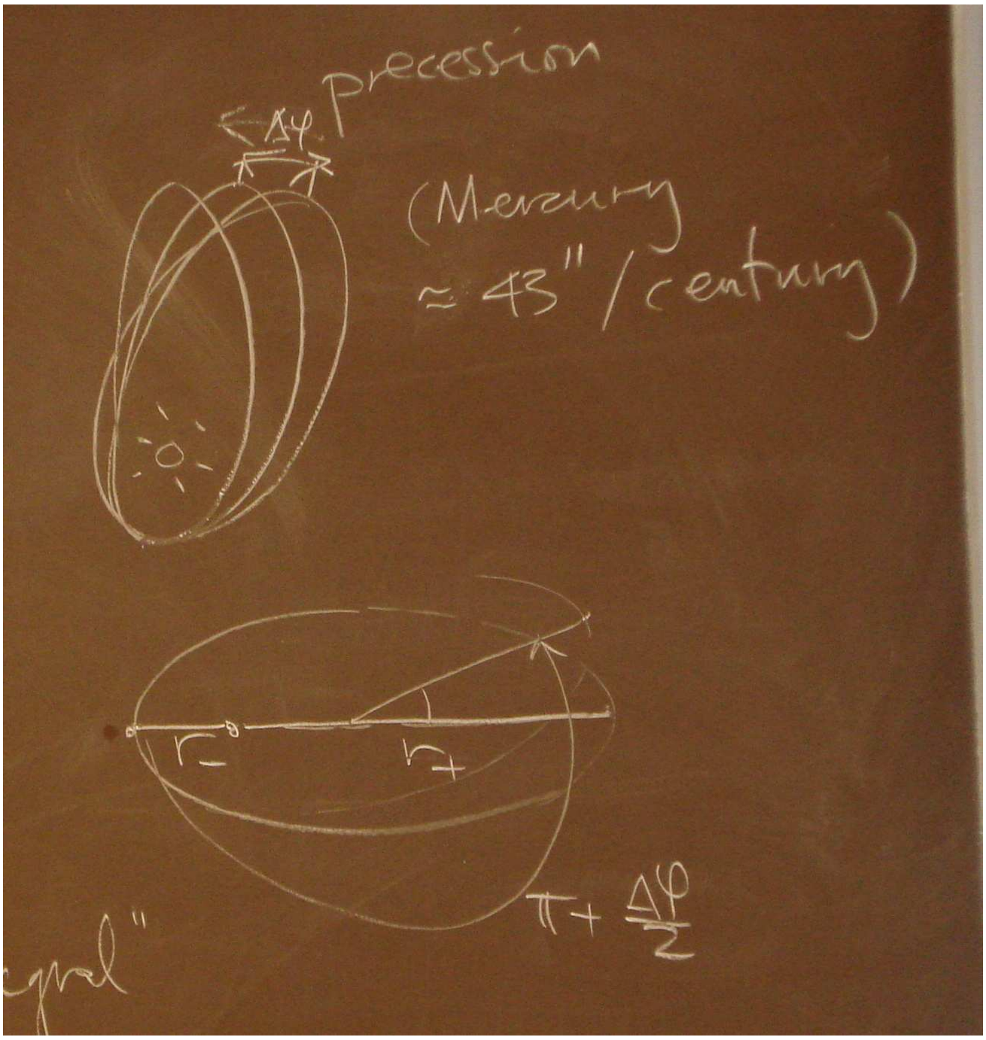

Figure 3. The precession angle $\Delta\varphi$. [The lower image is not quite correct. The angle is to be measured with respect to the central body, not the centre of the ellipse, as this picture makes it seem.]

Lowest order: $\Delta\varphi = 0$.

Next to lowest order:

$$\Delta\varphi = \frac{6\pi MG}{a(1-e^2)}$$

where a is the major semi-axis, and the minor semi-axis $b = a\sqrt{1-e^2}$. e is the eccentricity.

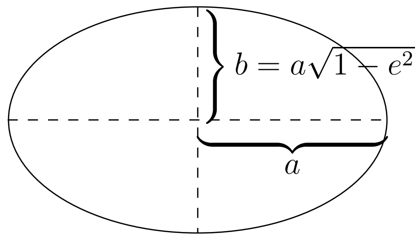

Figure 4. Ellipse.

For a planet with small eccentricity, $\Delta\varphi \approx 6\pi\ M\ G/a$. $\Delta\varphi \ll 1$ (or we have to go to higher order).

Symmetries ("isometries") "same metric"

Example: S^2 symmetric under 3-dimensional rotations. 3 isometries. S^D (D dimensional sphere): number of rotations in $D+1$ dimensions: $D\ (D+1)/2$. This is actually the maximal number of isometries you can have in D dimensions.

Example: Take flat D dimensional space. How may symmetries does it have? Rotations: $D(D-1)/2$. Translations: D. Together we have $D\,(D+1)/2$ isometries.

Example: Schwarzschild: Rotations: 3. Time translation: 1. $3+1=4$ isometries.

We count transformations that can be made infinitesimally small. Continuous transformations. Time reversal, parity are discrete symmetries. They are also symmetries, but we won't bother with them.

How do we check for (continuous) isometries?

- The metric should be "form-invariant", meaning that the new metric $g'_{\mu\nu}(x')$ should be the same functions of x' as the old metric $g_{\mu\nu}(x)$, before the coordinate transformation, was of x.

We are not saying that $g'_{\mu\cancel{\nu}}(x') = g_{\mu\cancel{\nu}}(x)$. We are saying $g'_{\mu\nu}(x) = g_{\mu\nu}(x)$.

Infinitesimally, $x'^{\mu} = x^{\mu} + \xi^{\mu}$

$$\delta g_{\mu\nu} \equiv g'_{\mu\nu}(x) - g_{\mu\nu}(x) = -2\,\mathrm{D}_{(\mu}\xi_{\nu)}$$

$$\boxed{\mathrm{D}_{(\mu}\,\xi_{\nu)} = 0}$$

If you want to check if the metric has symmetries, ask if there are vector fields that fulfil this. This is very restrictive. This is called the Killing equation. A solution $\xi_{\mu}(x)$ is called a Killing vector (field).