2008–11–20

5.1

"Find the equations of motion for a massive particle in Schwarzschild geometry."

The standard form of the metric for a static and isotropic gravitational field:

$$\mathrm{d}\tau^2 = -\,g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = B(r)\mathrm{d}t^2 - A(r)\,\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2$$

All the coefficients are functions of r, with no time dependence. In the case of a central mass M we solve Einstein's equations in empty space: $R_{\mu\nu} = 0$, and we get

$$B(r) = \frac{1}{A(r)} = 1 - \frac{2\,G\,M}{r}$$

The geometry is spherically symmetric. $\Rightarrow$ We only need to consider motion in the $\theta = \frac{\pi}{2}$ plane.

$$\mathrm{d}\tau^2 = B(r)\,\mathrm{d}t^2 - A(r)\,\mathrm{d}r^2 - r^2\mathrm{d}\varphi^2$$

Since the particle is massive we use τ to parametrise the trajectory. The Lagrangian is, with dot denoting derivative with respect to τ,

$$L = -\,g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = B(r)\,\dot{t}^2 - A(r)\dot{r}^2 - r^2\dot{\varphi}^2 = 1$$

Euler-Lagrange equations for t:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial L}{\partial t}\right) = 0 \quad \Rightarrow \quad \begin{cases} B(r)\,\dot{t} = \frac{1}{\sqrt{E}}, \text{for some constant } E \\ \mathrm{d}\tau = \sqrt{E}\,B(r)\,\mathrm{d}t \end{cases}$$

The choice of integration constant may seem peculiar at this time, but it will be expedient.

Euler-Lagrange equations for φ:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial L}{\partial \varphi}\right) = 0 \quad \Rightarrow \quad r^2\dot{\varphi} = C_1$$

$$\dot{\varphi} = \frac{\mathrm{d}\varphi}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{\mathrm{d}\varphi}{\mathrm{d}t}\frac{1}{\sqrt{E}\,B(r)} = \frac{C_1}{r_2}$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{C_1\sqrt{E}\,B(r)}{r^2} = \left[J = C_1\sqrt{E}\right] = \frac{J\,B(r)}{r^2}$$

$$r^2\frac{\mathrm{d}\varphi}{\mathrm{d}t} = J\,B(r)$$

Use $L = 1$:

$$1 = B(r)\,\dot{t}^2 - A(r)\,\dot{r}^2 - r^2\dot{\varphi}^2 = B(r)\,\frac{1}{E\,B(r)^2} - A(r)\left(\frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 - r^2\frac{C_1^2}{r^4} =$$

$$= \frac{1}{E\,B(r)} - A(r)\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2\frac{1}{E\,B(r)} - \frac{1}{r^2}\cdot\frac{J^2}{E}$$

$$\Rightarrow \quad E = \frac{1}{B(r)} - \frac{A(r)}{B(r)^2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 - \frac{J^2}{r^2}$$

Massless particle: We cannot use τ as our parameter.

$$L = \left(\frac{\mathrm{d}\tau}{\mathrm{d}\lambda}\right)^2 = 0$$

The equations are:

$$\begin{cases} r^2 \frac{\mathrm{d}\varphi}{\mathrm{d}t} = J B(r) \\ 0 = \frac{1}{B(r)} - \frac{A(r)}{B(r)^2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 - \frac{J^2}{r^2} \end{cases}$$

5.2

> "Using the Schwarzschild metric, find the proper length of the curves
>
> (a) $r = \theta = \text{const}, 0 \leqslant \phi \leqslant 2\pi$
>
> (b) $\theta = \phi = \text{const}, r_1 \leqslant r \leqslant r_2$.
>
> Comment on the result!"

Proper length in the Schwarzschild geometry ($\mathrm{d}s^2 = -\mathrm{d}\tau^2$)

$$s = \int \mathrm{d}s = \int_{\lambda_1}^{\lambda_2} \left(g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\right)^{1/2} \mathrm{d}\lambda$$

where the path $x^\mu(\lambda)$ through spacetime has been parametrised by λ.

a) $r = r_0, \theta = \theta_0, 0 \leqslant \varphi < 2\pi$. Parametrise using $\lambda = \varphi$.

$$\dot{t} = \dot{r} = \dot{\theta} = 0, \dot{\varphi} = 1$$

$$s = \int_0^{2\pi} \mathrm{d}\varphi \sqrt{g_{\varphi\varphi} \cdot 1 \cdot 1} = \int_0^{2\pi} \mathrm{d}\varphi \sqrt{\sin^2\theta_0 \, r_0^2} = 2\pi \, r_0 \sin\theta_0$$

This agrees with the result in the classical limit.

b) $\theta = \theta_0, \varphi = \varphi_0, r \leqslant r \leqslant r_2$.

Parametrise with $\lambda = r \Rightarrow \dot{t} = \dot{\theta} = \dot{\varphi} = 0, \dot{r} = 1$.

$$s = \int_{r_1}^{r_2} \mathrm{d}r \sqrt{g_{rr} \cdot 1 \cdot 1} = \int_{r_1}^{r_2} \mathrm{d}r \sqrt{A(r)} = \int_{r_1}^{r_2} \mathrm{d}r \frac{1}{\sqrt{1 - \frac{2MG}{r}}} =$$

$$= \int_{r_1}^{r_2} \mathrm{d}r \frac{\sqrt{r}\,\mathrm{d}r}{\sqrt{r - 2MG}} = \left[\begin{matrix}\text{Table of} \\ \text{integrals}\end{matrix}\right] = \left[\sqrt{r}\sqrt{r - 2MG} + 2MG \ln\left(\sqrt{r} + \sqrt{r - 2MG}\right)\right]_{r_1}^{r_2} =$$

$$= r_2\sqrt{1 - \frac{2MG}{r_2}} - r_1\sqrt{1 - \frac{2MG}{r_1}} + 2MG \ln\left(\frac{\sqrt{r_2}\left(1 + \sqrt{1 - 2MG/r_2}\right)}{\sqrt{r_1}\left(1 + \sqrt{1 - 2MG/r_1}\right)}\right)$$

The expression simplifies far from $r_\mathrm{s} = 2MG$, when $2MG/r_1 \ll 1$ and $2MG/r_2 \ll 1$. Then we can expand

$$\sqrt{1 - \frac{2MG}{r}} = 1 - \frac{1}{2}\left(\frac{2MG}{r}\right) + \cdots$$

$$\Rightarrow s = (r_2 - r_1) + 2MG \ln\left(\frac{-\sqrt{r_2}}{-\sqrt{r_1}}\right) + \mathcal{O}(MG)$$

$$= (r_2 - r_1) + MG \ln\left(\frac{r_2}{r_1}\right) + \mathcal{O}(MG)$$

$r_2 - r_1$ is the expected result in flat spacetime. $MG\ln(r_2/r_1)$ is the first order general relativistic correction.

5.3

> "Consider a massive particle in a space-time described by the Schwarzschild metric. For a given value of $j \equiv r^2\dot{\phi}$, is it possible to be in a circular orbit? If so, what is the value of r for this orbit?"

$$L = B(r)\,\dot{t}^2 - A(r)\,\dot{r}^2 - r^2\dot{\varphi}^2$$

($\theta = \pi/2$ by symmetry, $\dot{} = \frac{\mathrm{d}}{\mathrm{d}t}$)

Recall

$$\dot{t} = \frac{1}{\sqrt{E}\,B(r)} \tag{1}$$

$$r^2\dot{\varphi} = j \tag{2}$$

$$1 = \frac{1}{E\,B(r)} - A(r)\,\dot{r}^2 - \frac{j^2}{r^2} \tag{3}$$

For what values of j is it possible to have circular orbits in this metric, and what are the r for those orbits? (3) $\Rightarrow$

$$\dot{r}^2 = \frac{1}{E\,A(r)\,B(r)} - \frac{1}{A(r)}\left(1 + \frac{j^2}{r^2}\right) = \frac{1}{E} - \left(1 - \frac{2\,MG}{r}\right)\left(1 + \frac{j^2}{r^2}\right) =$$

$$= \frac{1}{E} - \left(1 - \frac{2\,MG}{r} + \frac{j^2}{r^2} - \frac{2\,MG\,j^2}{r^3}\right)$$

Differentiate with respect to τ:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(\dot{r}^2) = 2\,\dot{r}\,\ddot{r} = -\frac{\mathrm{d}r}{\mathrm{d}\tau}\frac{\mathrm{d}}{\mathrm{d}r}\left(-\frac{2\,MG}{r} + \frac{j^2}{r^2} - \frac{2\,MG\,j^2}{r^3}\right) = -\dot{r}\,\frac{\mathrm{d}}{\mathrm{d}r}(\ldots)$$

We can cancel $\dot{r} \Rightarrow$ [A student remarks that it is a bit dodgy to cancel $\dot{r}$ if we are looking for circular orbits, where $\dot{r} = 0$. Fredrik thinks that is not a problem, but is unsure of why.]

$$\ddot{r} = -\frac{\mathrm{d}}{\mathrm{d}r}\underbrace{\left(-\frac{MG}{r} + \frac{j^2}{2\,r^2} - \frac{MG\,j^2}{r^3}\right)}_{\equiv V_{\text{eff}}}$$

We can regard V_{eff} as an effective potential.

$$\ddot{r} = -\frac{\mathrm{d}}{\mathrm{d}r}V_{\text{eff}}(r)$$

Circular orbits:

$$\ddot{r} = 0 \Rightarrow \frac{\mathrm{d}}{\mathrm{d}r}V_{\text{eff}} = 0$$

If primes denote derivative with respect to r:

$$V_{\text{eff}}'(r) = \frac{MG}{r^2} - \frac{j^2}{r^3} + \frac{3\,MG\,j^2}{r^4} = 0$$

Since the Schwarzschild solution is only valid for $r > 0$, we can multiply by r^4:

$$r^2 MG - r\, j^2 + 3\, MG\, j^2 = 0$$

$$r^2 - r\frac{j^2}{MG} + 3\, j^2 = 0$$

$$r = \frac{j^2}{2\, MG} \pm \sqrt{\left(\frac{j^2}{2\, MG}\right)^2 - 3\, j^2}$$

$$r_\pm = \frac{j^2}{2\, MG}\left(1 \pm \sqrt{1 - 12\frac{M^2 G^2}{j^2}}\right)$$

What we really want to do is to find *stable* orbits, orbits where the orbiting body is not thrown out of the orbit by the slightest perturbation — since perturbations will always be present in a realistic situation. We are looking for a minimum of V_{eff}, not a maximum. Thus we require $V_{\text{eff}}'' > 0$.

$$r^5 V_{\text{eff}}'' = -2\, MG\, r^2 + 3\, j^2 r - 12\, MG\, j^2 =$$

$$\left[r^2 = \frac{j^2}{MG} r - 3\, j^2 \quad \text{at} \quad V_{\text{eff}}' = 0\right]$$

$$= -2\, MG\left(\frac{j^2}{MG} r - 3\, j^2\right) + 3\, j^2 r - 12\, MG\, j^2 = j^2(r - 6\, MG)$$

Finally, we study the orbits for the three cases.

i)

$$\frac{12 M^2 G^2}{j^2} > 1 \quad \Rightarrow \quad \text{No circular orbits (imaginary roots } r_\pm)$$

ii)

$$\frac{12\, M^2 G^2}{j^2} < 1 \quad \Rightarrow \quad \text{One stable orbit } (r_+), \text{ and one unstable orbit } (r_-)$$

iii)

$$\frac{12\, M^2 G^2}{j^2} = 1 \quad \Rightarrow \quad r_+ = r_- = r = \frac{j^2}{2\, MG} = 6\, MG$$

Only a single orbit is possible. V_{eff} has a saddle point.

Classical limit: $2\, MG \ll j$

$$r_+ = \frac{j^2}{2\, MG}(1 + 1) = \frac{j^2}{MG}$$

$$r_- = \frac{j^2}{2\, MG}\left(1 - \left(1 - \frac{1}{2} \cdot 12\frac{M^2 G^2}{j^2}\right)\right) = 3\, MG$$

Usually $r_- = 3\, M\, G = \frac{3}{2}\, r_{\text{s}}$ is inside the massive object. The Schwarzschild solution only applies to free space, so if we really want to do this inside an object, we would have to consider $R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} R = -8\, \pi\, G\, T_{\mu\nu}$.

In the classical limit there is *one* stable orbit at $r_+ = j^2 / MG$.