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Today we will discuss action. We will talk a bit more about general coordinate transformations, and maybe an alternative way of understanding them and what they do.

Action principle for gravity?

An action is a functional. It is some function, some field, and put it into a box, and out comes a number — that is a functional. A functional is a function that takes a function and maps it to a number. So an action is a functional on the space of field configurations. But it is not just any functional.

Say we have some field $\phi(x)$ (in our case, ultimately, we will treat the field $g_{\mu\nu}(x)$ — the metric itself).

$$\phi(x) \longrightarrow \boxed{\ S\ } \longrightarrow S[\phi] \in \mathbb{R}$$

The action is denoted $S[\phi]$, with square brackets being standard notation for functionals. The action is a functional on the space of field configurations *whose extrema are solutions of field equations*.

Example: Classical mechanics:

$$S = \int \mathrm{d}t\,(T - V) = \int \mathrm{d}t\left(\frac{1}{2}m\,\dot{x}^2 - V(x)\right)$$

$$S[x+\varepsilon] = \int \mathrm{d}t\left(m\,\dot{x}\,\dot{\varepsilon} - \frac{\mathrm{d}V}{\mathrm{d}x}\,\varepsilon\right) = \int \mathrm{d}t\,\varepsilon(t)\underbrace{\left[-m\,\ddot{x} - \frac{\mathrm{d}V}{\mathrm{d}x}\right]}_{=:\frac{\delta S}{\delta x(t)}}$$

This is more or less the definition of the functional derivative.

OK, that was just a reminder. In our case, we want a similar expression for the field $g_{\mu\nu}$:

$$S[g] = \text{constant} \times \int \mathrm{d}^4x\,\sqrt{|g|}\,(\quad ? \quad)$$

What to write here? It should be a scalar. If it does not transform covariantly, I don't expect the equations that come out to be covariant. So it is a scalar. We also know that Einstein's equations contain two derivatives. How much choice do we have? We can choose the curvature scalar R.

$$S[g] = k \int \mathrm{d}^4x\,\sqrt{|g|}\,R$$

We could also enter a constant in addition to R, and that way we can enter the cosmological constant into the equations.

What about the constant k? $S[g]$ has dimension $M\,L$, since $S/\hbar$ is dimensionless (compare path integrals in quantum mechanics). R has dimension L^2, and d^4x has dimension L^4. The integral is thus L^2. $[G] = M^{-1}\,L$. So a $1/G$ would be appropriate.

$$S[g] = -\frac{1}{8\pi G}\int \mathrm{d}^4x\,\sqrt{|g|}\,R$$

This is what we will want right now. We will see why this is a good thing. This is our candidate for an action for gravity. Now it is time to see how it works. We want to take the functional derivative of this candidate action with respect to the metric field $g_{\mu\nu}(x)$. We have to be careful, since the integration measure contains the metric as well, in $\sqrt{|g|}$.

I remind you of how curvature is defined:

$$-R_{\mu\nu}{}^{\rho}{}_{\sigma} = (\partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu + \Gamma_\mu \Gamma_\nu - \Gamma_\nu \Gamma_\mu)^{\rho}{}_{\sigma}$$

$$-\delta R_{\mu\nu}{}^{\rho}{}_{\sigma} = (\partial_\mu \delta\Gamma_\nu - \partial_\nu \delta\Gamma_\mu + \delta\Gamma_\mu \Gamma_\nu + \Gamma_\mu \delta\Gamma_\nu - \delta\Gamma_\nu \Gamma_\mu - \Gamma_\nu \delta\Gamma_\mu)^{\rho}{}_{\sigma}$$

$$\partial_\mu \delta\Gamma_\nu + \Gamma_\mu \delta\Gamma_\nu - \delta\Gamma_\nu \Gamma_\mu = \partial_\mu \delta\Gamma_\nu + [\Gamma_\mu, \delta\Gamma_\nu]$$

$[\Gamma_\mu, \delta\Gamma_\nu]$ is the two terms in $\mathrm{D}_\mu \delta\Gamma_\nu$ where Γ acts on the two “invisible” indices.

$$-\delta R_{\mu\nu}{}^{\rho}{}_{\sigma} = (\mathrm{D}_\mu \delta\Gamma_\nu - \mathrm{D}_\nu \delta\Gamma_\mu)^{\rho}{}_{\sigma}$$

The difference of two connections is always a tensor.

We are not interested in the Riemann curvature tensor in itself, so go to the Ricci tensor:

$$-\delta R_{\nu\sigma} = \mathrm{D}_\mu \delta\Gamma^{\mu}_{\nu\sigma} - \mathrm{D}_\nu \delta\Gamma^{\mu}_{\mu\sigma}$$

Now we could start talking about the variation of Γ in terms of the variation of g, but we don't need that. We will see that in a moment.

$$\sqrt{|g|}\, R = \sqrt{|g|}\, g^{\mu\nu} R_{\mu\nu}$$

$$\delta g^{-1} = -g^{1}\, \delta g\, g^{-1}$$

Do you recognise this last expression? It is very simple: $\delta(g\, g^{-1}) = 0 = \delta g\, g^{-1} + g\, \delta g^{-1}$. This is a handy formula.

Now we have to take the variation of the determinant.

$$\delta(\det g) = \det g\, \mathrm{tr}(g^{-1} \delta g)$$

One can show this in several ways. $\exp(\mathrm{tr}\, M) = \det(\exp(M))$. (Easy to see if M is diagonal.)

$$\delta\sqrt{\det g} = \frac{1}{2}\sqrt{\det g}\, \mathrm{tr}\big(g^{-1}\, \delta g\big)$$

Now we are ready to take the variation of S:

$$\delta S = -\frac{1}{16\pi G}\int \mathrm{d}^4 x \bigg\{ \frac{1}{2}\sqrt{|g|}\, g^{\rho\sigma} \delta g_{\rho\sigma}\, g^{\mu\nu} R_{\mu\nu} -$$

$$-\sqrt{|g|}\, g^{\mu\rho}\, \delta g_{\rho\sigma}\, g^{\sigma\nu} R_{\mu\nu} + \sqrt{|g|}\, g^{\mu\nu}\big(-\mathrm{D}_\rho \delta\Gamma^{\rho}_{\mu\nu} + \mathrm{D}_\mu\, \delta\Gamma^{\rho}_{\nu\rho}\big)\bigg\}$$

This last thing is the integral of a divergence:

$$\sqrt{|g|}\, \mathrm{D}_\rho V^\rho \text{ where } V^\rho = -g^{\mu\nu} \delta^{\rho}_{\mu\nu} + g^{\rho\nu} \delta\Gamma^{\mu}_{\mu\nu}$$

$$\mathrm{D}_\rho V^\rho = \frac{1}{\sqrt{|g|}}\, \partial_\rho\Big(\sqrt{|g|}\, V^\rho\Big)$$

So $\sqrt{|g|}\mathrm{D}_\rho V^\rho$ is just an ordinary divergence, and gives rise to boundary terms, which we in turn ignore. So the entire thing that came from the variation of the Ricci tensor goes away, without us ever needing to consider what $\delta\Gamma$ really is in terms of the metric.

$$\delta S = -\frac{1}{8\pi G}\int \mathrm{d}^4\sqrt{g}\,\delta g_{\rho\sigma}\left\{-R^{\rho\sigma}+\frac{1}{2}\,g^{\rho\sigma}R\right\}$$

Action principle $\Rightarrow$

$$R_{\mu\nu}-\frac{1}{2}\,g_{\mu\nu}\,R=0.$$

This is what Hilbert realised. The action is called Einstein-Hilbert action.

If we want matter to interact with the field:

$$S=-\frac{1}{8\pi G}\int \mathrm{d}^4x\,\sqrt{|g|}\,R+S_{\text{matter}}[\phi,g]$$

$$\frac{1}{\sqrt{|g|}}\cdot\frac{\delta S}{\delta g_{\mu\nu}(x)}=\frac{1}{8\pi G}\left(R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R\right)+\frac{1}{\sqrt{|g|}}\cdot\frac{\delta S_{\text{matter}}}{\delta g_{\mu\nu}}$$

If we want $R_{\mu\nu}-\frac{1}{2}\,g_{\mu\nu}\,R=-8\pi G\,T_{\mu\nu}$ to hold, we must have

$$\frac{1}{\sqrt{|g|}}\cdot\frac{\partial S_{\text{matter}}}{\partial g_{\mu\nu}}=T^{\mu\nu}$$

("Gauge symmetry"): general coordinate transformations (diffeomorphisms). (Diffeomorphisms is, at least for a physicist, the same thing as coordinate transformations.) Consider an infinitesimal change $x'^\mu=x^\mu+\varepsilon^\mu(x)$. Everything depends on the transformation matrix

$$M^\mu{}_\nu=\frac{\partial x'^\mu}{\partial x^\nu}=\delta^\mu_\nu+\partial_\nu\,\varepsilon^\mu$$

$$\left(M^{-1}\right)^\mu{}_\nu=\delta^\mu{}_\nu-\partial_\nu\varepsilon^\mu$$

$$g'_{\mu\nu}(x')=\left(\delta^\rho_\mu-\partial_\mu\varepsilon^\rho\right)\left(\partial^\sigma_\nu-\partial_\nu\,\varepsilon^\sigma\right)\,g_{\rho\sigma}(x)$$

I don't change the coordinates, and ask how the field changes. [Active view of the transformation.]

$$g'_{\mu\nu}(x')\simeq g'_{\mu\nu}(x)+\varepsilon^\rho\partial_\rho\,g'_{\mu\nu}(x)$$

The difference between $g_{\mu\nu}$ and $g'_{\mu\nu}$ is of order ε, so to order ε we can remove that prime:

$$g'_{\mu\nu}(x)\simeq g'_{\mu\nu}(x)+\varepsilon^\rho\partial_\rho g_{\mu\nu}(x)$$

$$\delta g_{\mu\nu}(x)=g'_{\mu\nu}(x)-g_{\mu\nu}=-\varepsilon^\rho\partial_\rho g_{\mu\nu}-\partial_\mu\varepsilon^\rho\,g_{\nu\rho}-\partial_\nu\varepsilon^\rho g_{\mu\rho}$$

"This looks completely stupid, but it is not."

$$\delta g_{\mu\nu}(x)=-\varepsilon_\sigma\,g^{\sigma\rho}\partial_\rho g_{\mu\nu}-\partial_\mu\varepsilon_\nu+\varepsilon_\sigma\,g^{\rho\sigma}\,\partial_\mu g_{\nu\rho}-\partial_\nu\varepsilon_\mu+\varepsilon_\sigma\,g^{\rho\sigma}\,\partial_\nu g_{\mu\rho}=$$

$$=-\partial_\mu\varepsilon_\nu-\partial_\nu\varepsilon_\mu+2\Gamma^\rho_{\mu\nu}\,\varepsilon_\rho=-\mathrm{D}_\mu\varepsilon_\nu-\mathrm{D}_\nu\varepsilon_\mu=-2\,\mathrm{D}_{(\mu}\varepsilon_{\nu)}$$

$$\boxed{\delta g_{\mu\nu}(x)=-2\mathrm{D}_{(\mu}\varepsilon_{\nu)}}$$

Take this variation and plug into the action and see what happens:

$$\delta S =$$

The action should not depend on my choice of coordinates. This is a symmetry of the theory. We must get $\delta S = 0$.

$$\delta S = -\frac{1}{8\pi G}\int \mathrm{d}^4x\,\sqrt{|g|}(-2\mathrm{D}_\mu\varepsilon_\nu)\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right) + \int \mathrm{d}^4x\sqrt{|g|}(-2\mathrm{D}_\mu\varepsilon_\nu)\,T^{\mu\nu}$$

[“The world is made for me to make sloppy partial integration”].

$$\mathrm{D}_\mu T^{\mu\nu} = 0$$

This is how coordinate transformation leads to the conservation of a current.

$$S = \int \mathrm{d}^4x\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + A_\mu J^\mu\right)$$

What happens when you vary A_μ? This gives you Maxwell’s equations. $\delta A_\mu = \partial_\mu\Lambda$ gives you conservation of current. Gauge symmetry $\longleftrightarrow$ conserved current.