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Today we will talk about the Schwarzschild solution. There are some more aspects of Einsteins equations that we will also treat, perhaps today.

The Schwarzschild solution describes a static gravity field (geometry) around a spherically symmetric body.

For the astronomy of the solar system we know that there is a Newtonian limit, and we know something experimentally about the accuracy of Newton's description $\Phi = -\,M\,G/r$. We know beforehand, when we solve Einstein's equations, that we will get something that looks like $\Phi = -\,M\,G/r$.

By solving Einstein's equations, we mean finding $g_{\mu\nu}(x)$. We know that the solution is not unique, we can always make coordinate transformations. We will make an ansatz, and see if we can solve the resulting differential equation. We solve it in empty space, outside of the massive body. The mass will enter as an arbitrary integration constant. $T_{\mu\nu} = 0$. The tricky thing is to use the freedom to choose coordinates in a suitable way to simplify your ansatz, and use the assumption of symmetry.

- Use symmetry.
- Use the freedom to choose coordinates. (Make a "gauge choice".)

Let us call the coordinates $t, \boldsymbol{r}$. First of all, we say *static*: in the metric, nothing will depend on t. It must be possible to choose a coordinate system that does not depend on t. (Staticity is also a symmetry. We will later gain better tools to identify symmetries. Now we just use it.) $g_{\mu\nu}(\boldsymbol{r})$: no dependence on t.

When constructing $\mathrm{d}s^2$, use only $r = \sqrt{\boldsymbol{r}\cdot\boldsymbol{r}}$, $\boldsymbol{r}\cdot\mathrm{d}\boldsymbol{r}$, $\mathrm{d}\boldsymbol{r}\cdot\mathrm{d}\boldsymbol{r}$ (spherically symmetric). The only thing the functions in the metric can depend on is r. Ansatz:

$$\mathrm{d}s^2 = -\,a(r)\,\mathrm{d}t^2 + b(r)\,\mathrm{d}r^2 + c(r)\,\mathrm{d}t\,\boldsymbol{r}\cdot\mathrm{d}\boldsymbol{r} + d(r)\,(\boldsymbol{r}\cdot\mathrm{d}\boldsymbol{r})^2$$

This is not the end of the story. We can still simplify this by making nice coordinate transformations. We have only said that nothing depends on time, but we haven't fixed what time is. If we go to another time coordinate $t' = t + g(r)$

$$\mathrm{d}t = \mathrm{d}t' - \partial_i\, g(r)\,\mathrm{d}r^i$$

We use this freedom to get rid of $c(r)$. So, with suitably defined time t:

$$\mathrm{d}s^2 = -\,a(r)\,\mathrm{d}t^2 + b(r)\,\mathrm{d}r^2 + d(r)\,(\boldsymbol{r}\cdot\mathrm{d}\boldsymbol{r})^2$$

Instead of using $b(r), d(r)$, we use functions where

$$\mathrm{d}s^2 = -\,B(r)\,\mathrm{d}t^2 + A(r)\,\mathrm{d}r^2 + C(r)\,\underbrace{\left(\mathrm{d}\theta^2 + \sin^2\!\theta\;\mathrm{d}\varphi^2\right)}_{\mathrm{d}\Omega^2}$$

where $\mathrm{d}^2\Omega$ is the metric on unit S^2. There is one more thing we can do: rescale r.

$$r' = h(r)$$

There is complete freedom in choosing r, apart from global properties (if we want r to go from 0 to ∞, we wouldn't want an oscillating function, for instance). We use $r' = h(r)$ to set e.g. $C(r) = r^2$. Here we fix what we mean by radius, r. We relate the radial coordinate to the area of the two-sphere. (An alternative choice, which would be nice but won't be used here, is $C(r) = r^2 A(r)$. That would give us something like $-\,B(r) + A(r)\big(r^2 + r^2\mathrm{d}\Omega^2\big)$)

So in the end, our ansatz will be

$$\mathrm{d}s^2 = -B(r)\,\mathrm{d}t^2 + A(r)\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2$$

That is as simple as it gets.

From $\mathrm{d}s^2$, from $g_{\mu\nu}$, we form the affine connection, and the Riemann tensor, or rather the Ricci tensor:

$$R_{\mu\nu} = 0 \quad \text{(Einstein's equations in empty space)}$$

$$g_{\mu\nu} \to \Gamma \to R_{\mu\nu}{}^{\rho}{}_{\sigma} \to R_{\mu\nu} \quad \text{or} \quad g_{\mu\nu} \to \Gamma \to R_{\mu\nu} \text{ directly.}$$

$$\Gamma^r_{rr} = \frac{A'}{2A}, \quad \Gamma^r_{\theta\theta} = -\frac{r}{A}, \quad \Gamma^r_{\varphi\varphi} = -\frac{r\sin^2\theta}{A}, \quad \Gamma^r_{tt} = \frac{B'}{2A}, \quad \Gamma^r_{r\theta} = \frac{1}{r}, \quad \Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta,$$

$$\Gamma^\varphi_{r\varphi} = \frac{1}{r}, \quad \Gamma^\varphi_{\theta\varphi} = \cot\theta, \quad \Gamma^t_{rt} = \frac{B'}{2B}$$

Now we need some... coffee.

Do the calculation! The Ricci tensor turns out to be diagonal:

$$R_{tt} = -\frac{B''}{2A} + \frac{1}{4}\frac{B'}{A}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{1}{r}\frac{B'}{A} = 0$$

$$R_{rr} = \frac{B''}{2B} - \frac{1}{4}\frac{B'}{B}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{1}{r}\frac{A'}{A} = 0$$

$$R_{\theta\theta} = -1 + \frac{r}{2A}\left(-\frac{A'}{A} + \frac{B'}{B}\right) + \frac{1}{A} = 0$$

$$R_{\varphi\varphi} = \sin^2\theta\, R_{\theta\theta} = 0$$

Three equations (not counting the one for $R_{\varphi\varphi}$) for two functions. It will be a consistency check in the end, that it works out.

$$\frac{R_{tt}}{B} + \frac{R_{rr}}{A} = -\frac{1}{rA}\left(\frac{B'}{B} + \frac{A'}{A}\right) = 0$$

$$(\log A\,B)' = \frac{B'}{B} + \frac{A'}{A} = 0$$

$$A\,B = \text{constant}$$

At infinity, we would like this to reduce to Minkowski space. That means $A \to 1$, $B \to 1$ as $r \to \infty$. So we take $A\,B = 1$.

$$R_{tt} = -\frac{B''}{2A} + \frac{1}{4}\frac{B'}{A}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{1}{r}\frac{B'}{A} = 0$$

$$R_{rr} = \frac{B''}{2B} - \frac{1}{4}\frac{B'}{B}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{1}{r}\frac{A'}{A} = 0$$

$$R_{\theta\theta} = -1 + \frac{r}{2A}\left(-\frac{A'}{A} + 2\frac{B'}{B}\right) + \frac{1}{A} = 0$$

$$0 = R_{\theta\theta} = -1 + r\,B' + B$$

$$(r\,B)' = 1 \quad \Rightarrow \quad r\,B = \gamma + r \quad \Rightarrow \quad B = 1 + \frac{\gamma}{r} \quad A = \frac{1}{1 + \frac{\gamma}{r}}$$

(Check that R_{rr} or R_{tt} gives the same equation.)

We get an integration constant γ, which is quite natural. The mass of the central body has to enter somewhere. As $r \to \infty$:

$$B = 1 + \frac{\gamma}{r}$$

$$B = -g_{00} \simeq 1 + 2\,\Phi = 1 - 2\frac{M\,G}{r} \quad \text{in the Newtonian limit}$$

$$\gamma = -2\,M\,G$$

Schwarzschild solution:

$$\mathrm{d}s^2 = -\left(1 - \frac{2\,M\,G}{r}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - \frac{2\,M\,G}{r}} + r^2\,\mathrm{d}\Omega^2$$

- How does a particle/light ray move in this geometry?
- Large distances: $r \gg M\,G$ — Newton.

Planet trajectories are a bit different, compared to the Newtonian limit. Instead of perfectly elliptical orbits, we will get something that is close to an ellipse, but not quite: the perihelion *precesses*. (The precession of Mercury's orbit was well known even before general relativity.)

For the surface of the sun, or the earth, $2M\,G/r$ is very small. But there is in principle nothing stopping it from being large — even larger than one, really.

The earth has a radius $R \gg 2\,M\,G$. But let us consider an object where $r = 2\,M\,G$ is still outside the object. Then, at this distance, the metric becomes singular. We have to investigate whether this is a physical singularity, where something goes wrong with the physical theory itself. Here, that is not the case: this is a coordinate singularity. We could choose coordinates where this singularity does not exist. If this were a real singularity, the curvature, for instance, should go to infinity or something. If the components of $R_{\mu\nu}{}^{\rho}{}_{\sigma}$ are singular that does not really say much, but we can study a scalar: $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = 48\,M^2\,G^2/r^6$. This is completely regular at $r = 2\,M\,G$, but it is singular in the middle.

(If $r = 2\ M\ G$ lies outside the object, we are dealing with a very dense object. Not even a neutron star is small enough. This will be a black hole.)

$t = v - 2\,M\,G \log|r/2\,M\,G - 1| - r$

$$\mathrm{d}s^2 = -B(r)\mathrm{d}v^2 - 2\,\mathrm{d}v\,\mathrm{d}r - r^2\mathrm{d}\Omega^2$$

Here the apparent singularity at $r = 2\,M\,G$ is not there.

$$\begin{array}{lccc} \mathrm{d}s^2 = & -\left(1 - \frac{2\,M\,G}{r}\right)\mathrm{d}t^2 & +\frac{\mathrm{d}r^2}{1 - \frac{2\,M\,G}{r}} & + \cdots \\ r > 2\,M\,G & - & + & \\ r < 2\,M\,G & + & - & \end{array}$$

The distance $r = 2\,M\,G$ is called the *horizon*. At $r = 0$, however, we have a real singularity.

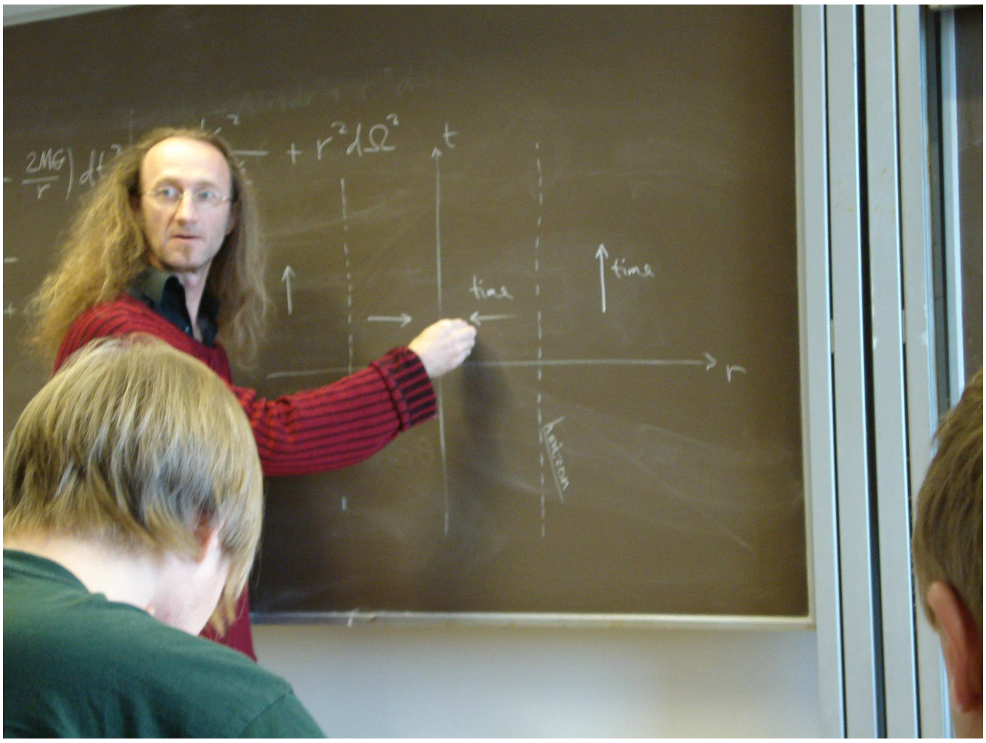

Figure 1. Outside the horizon, t takes the role of the time coordinate. Forward in time $\Rightarrow$ increasing t. At the horizon, the Schwarzschild coordinates do not apply. Other coordinate systems are more suitable there. Inside the horizon, r takes the role of the time coordinate, while t becomes a spatial coordinate. Inside the horizon, forward in time $\Rightarrow$ decreasing r.