2008–11–11

We are a little ahead of schedule, which is good. That gives us a little more time to consider interesting physics, once we get through the mathematical formalism.

We will continue using curvature today, and write down Einstein's equations. Reminder: Gravity is about two things: 1) How does gravity affect objects that move around (geodesic equation), and 2) How does the energy and momentum that moves around affect gravity? We have previously reasoned about why energy and momentum have to be the source of the gravitational field. Most of the course (apart from the part about symmetry) will be spent studying solutions to Einstein's equations.

We remind ourselves of the curvature tensor (Riemann tensor):

$$-R_{\mu\nu}{}^{\rho}{}_{\sigma}V^{\sigma} = [\mathrm{D}_{\mu}, \mathrm{D}_{\nu}]V^{\rho}$$

(Remember that no derivatives will act on V^{ρ} when we simplify this expression.)

$$-R_{\mu\nu}{}^{\rho}{}_{\sigma} = [\mathrm{D}_{\mu}, \mathrm{D}_{\nu}]^{\rho}{}_{\sigma}$$

[“Why do you have other courses besides this one? Thou shalt not...”]

Weinberg writes $-R^{\rho}_{\sigma\mu\nu}$, but that is the same thing, because of the symmetries we showed last time:

$$R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma} = R_{[\mu\nu]\rho\sigma} = R_{\mu\nu[\rho\sigma]}$$

We also had $R_{\mu[\nu\rho\sigma]} = 0$. These are the algebraic properties of the curvature tensor.

Compare with the case of electrodynamics: the corresponding tensor that tells you everything about how strong the field is, that is the field strength $F_{\mu\nu} = F_{[\mu\nu]}$ (it is antisymmetric). For those who have taken advanced classical physics: $\mathrm{D}_{\mu} = \partial_{\mu} + A_{\mu}$, and we got $[\mathrm{D}_{\mu}, \mathrm{D}_{\nu}] = F_{\mu\nu} = \partial_{\nu}A_{\mu} - \partial_{\nu}A_{\mu}$. A_{μ} is called the gauge potential or the connection. In general relativity, we have the affine connection instead. When we know that the field strength is constructed like this, we know one differential symmetry as well: $\partial_{[\mu}F_{\nu\lambda]} = 0$. (These are two of Maxwell's equations, those without charges or currents.) We call this the Bianchi identity. We get a Bianchi identity in general relativity too:

$$-\mathrm{D}_{\lambda}R_{\mu\nu}{}^{\rho}{}_{\sigma} = \mathrm{D}_{\lambda}[\mathrm{D}_{\mu}, \mathrm{D}_{\nu}]$$

This is slightly misleading. Normally differential operators act on everything to the right of them. “(∂f)” $= [\partial, f] = \partial f - f\partial$.

$$-\mathrm{D}_{\lambda}R_{\mu\nu}{}^{\rho}{}_{\sigma} = [\mathrm{D}_{\lambda}, [\mathrm{D}_{\mu}, \mathrm{D}_{\nu}]]^{\rho}{}_{\sigma}$$

Now we can think of it with the derivatives acting on everything to the right of them, because the extra terms are explicitly subtracted.

$$\Rightarrow \mathrm{D}_{[\lambda}R_{\mu\nu]}{}^{\rho}{}_{\sigma} = 0$$

$$[A, [B, C]] + \text{cyclic permutations} = ... = [\text{Jacobi}] = 0$$

The Jacobi identity is trivial, one just has to write everything out.

$\mathrm{D}_{[\lambda}R_{\mu\nu]}{}^{\rho}{}_{\sigma} = 0$ is the Bianchi identity for general relativity. It is very important.

We are looking for Einstein's equations. We want them to be second order field equations for the metric $g_{\mu\nu}$), and we also want them to reduce to Newtonian gravity in a suitable limit. In electrodynamics we have $\partial_\beta F^{\alpha\beta} = J^\alpha$ (in Heaviside-Lorentz units with $c = 1$). Current is conserved ($\partial_\alpha J^\alpha = 0$). So $\partial_\alpha \partial_\beta F^{\alpha\beta} = 0$ which can be seen through the antisymmetry of the indices of $F^{\alpha\beta}$.

We want something constructed from the curvature to be $T_{\mu\nu}$, $(R)_{\mu\nu} \sim T_{\mu\nu}$, since the number of derivatives match. Some tensor $G_{\mu\nu}$, linear in the curvature, gives:

$$G_{\mu\nu} = k\, T_{\mu\nu}$$

We want current to be conserved $\mathrm{D}^\nu T_{\mu\nu} = 0$. We need the left hand side to fulfil the same equation, we need $\mathrm{D}^\nu G_{\mu\nu} = 0$. The Bianchi identity has derivatives of the curvature tensor, but with far too many indices. We take it and contract two indices. Multiplying the Bianchi identity by δ^μ_ρ:

$$0 = \delta^\mu_\rho \big(\mathrm{D}_\mu R_{\nu\lambda}{}^\rho{}_\sigma + 2\, \mathrm{D}_{[\nu} R_{\lambda]\mu}{}^\rho{}_\sigma\big) = \mathrm{D}_\rho R_{\nu\lambda}{}^\rho{}_\sigma - 2\, \mathrm{D}_{[\nu} R_{\lambda]\sigma}$$

Now we have the Ricci tensor here. (Typical physicist's thing to do, denoting different things by the same symbol. Mathematicians would call the Ricci tensor Ric or something like that.)

$$\ldots \times g^{\nu\sigma}: \quad 0 = -\,\mathrm{D}^\rho R_{\lambda\rho} - \mathrm{D}^\sigma R_{\lambda\sigma} + \mathrm{D}_\lambda R = \mathrm{D}^\rho(-2\, R_{\lambda\rho} + g_{\lambda\rho} R)$$

$$\mathrm{D}^\nu \left(R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R \right) = 0$$

Now we have something that we could use for the left hand side, for $G_{\mu\nu}$. $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} R$ is referred to as the "Einstein tensor". Einstein's equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = k\, T_{\mu\nu}$$

for some constant k. This constant has yet to be determined. We will determine this using the only other thing we need to check: the Newtonian limit. Let us first do a dimensional analysis. Since $c = 1$, $L = T$, $M = E$ (length and time are the same dimension, mass and energy have the same dimension). Had we been doing quantum mechanics with $\hbar = 1$, we would have $L = M^{-1}$, but we are not doing quantum mechanics.

Normally we would think of coordinates as having dimension length $[x] = L$. We have seen examples where this has not been the case, e.g., the angle coordinate in polar coordinates. We shall ignore this difficulty and consider coordinates of dimension length. $[g] = 1$. This means that $[R] = L^{-2}$. We have $T_{00} = \rho$, what dimension does that have?

$$[T_{00}] = M\, L^{-3} = [T_{\mu\nu}]$$

This gives us the dimension of the k we are looking for: $[k] = M^{-1} L$. Suspecting that Newton's constant G will enter somewhere, we look for the dimension of G. $[G] = ?$

$$\Phi = -\,\frac{m\, G}{r}$$

$$[\Phi] = 1$$

$$\Rightarrow [G] = M^{-1}\, L$$

Now it seems natural that $k \propto G$. It can't really be anything else.

Newton (static mass distribution):

$$\nabla^2\Phi = 4\pi G\,\rho$$

(To see the 4π, take a spherical body and integrate to get Newtonian expression for the gravitational force. Compare how we get Coulomb's law from Maxwell's equations, and note how the 4π enter there.)

Static mass distribution: $T_{00} = \rho, \quad T_{0i} = 0, \quad T_{ij} = 0.$

$$R_{00} - \frac{1}{2}\,g_{00}\,R = k\,T_{00}, \quad R_{ij} - \frac{1}{2}g_{ij}R = k\,T_{ij} = 0$$

Weak fields: we can linearize. $g_{00} = -1 - 2\Phi$, $\Phi \ll 1$. We throw away anything that is higher than first order in Φ. Thus we can replace $g_{00} \to -1$ and $g_{ij} \to \delta_{ij}$:

$$R_{00} + \frac{1}{2}\,R = k\,T_{00}, \quad R_{ij} - \frac{1}{2}\,\delta_{ij}R = k\,T_{ij} = 0$$

$$R \approx -R_{00} + R_{ii} = -R_{00} + \frac{1}{2}\,\delta_{ii}\,R = -R_{00} + \frac{3}{2}\,R \quad \Rightarrow R = 2\,R_{00}$$

$$G_{00} = 2\,R_{00}$$

Now we have shown that

$$2\,R_{00} = k\,T_{00}$$

$$R_{00} = -R_{\mathbf{0000}} + R_{\mathbf{0}i\mathbf{0}i}$$

$$R_{\mu\nu\rho\sigma} \approx \frac{1}{2}\,(\partial_\mu\partial_\rho g_{\nu\sigma} + 3\,\mathrm{terms})$$

Static $\partial_0(...) = 0$.

$$R_{0000} = 0$$

$$R_{i0j0} = \frac{1}{2}\,\partial_i\partial_j g_{00}$$

$$R_{00} = \frac{1}{2}\nabla^2 g_{00}$$

$$G_{00} = \nabla^2 g_{00} = -2\,\nabla^2\Phi$$

$$G_{00} = k\,T_{00} \quad \Rightarrow \quad -2\nabla^2\Phi = k\,\rho$$

$$\Rightarrow k = -8\pi G$$

And that's it. That is maybe the most important point in this course.

$$\boxed{R_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}R = -8\pi G\,T_{\mu\nu}}$$

The minus sign depends on the sign convention for the Riemann tensor.

We could add to one side of this equation a constant times $g_{\mu\nu}$. This is called the cosmological constant. We will use it when we consider cosmology, but we will forget about it for the moment.

Solve!

We can't solve it in general. We will solve it in some specific cases. For example, the static and spherically symmetric situation. That gives us the Schwarzschild solution, and the gravitational field around a star. (It gives us black holes, too.)

We will also study the weak field limit and find gravitational waves.

We will solve, in some special situations, the universe. (Using a highly symmetric ansatz.)

The best part of the course, according to the lecturer, is when we insert the entire universe into the equations.

The equations are very non-linear, not even polynomial (it contains the inverse metric). It is astonishing that you can even do these things. This non-linearity is a great difference when comparing to electromagnetism. Gravity self-interacts. There is no superposition principle in these equations.

How unique is a given solution? It is unique up to coordinate transformations. This is the great symmetry that governs general relativity. We can call it a gauge symmetry.

Taking the analogy with electromagnetism again: The solution $A_\mu(x)$ is unique up to ...? $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$. $A'_\mu = A_\mu + \partial_\mu\Lambda(x)$. This is called a gauge transformation, an arbitrariness of A_μ. Possible conditions (gauge choices):

•$A_0 = 0$ (static gauge).

•$\partial^\mu A_\mu = 0$ (Lorentz gauge).

We know how $g_{\mu\nu}$ transforms:

$$g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\, g_{\rho\sigma}(x)$$

This can be brought to a form more reminiscent of $A'_\mu = A_\mu + \partial_\mu\Lambda(x)$, by considering infinitesimal coordinate transformations. (Will probably be done on Monday.)