2008–11–06

Exercises 2.1, 2.2 and 3.1

> "Find the metric for a flat two-dimensional surface expressed in polar coordinates."
> (Problem 2.1)
>
> "Calculate the affine connection $\Gamma^{\lambda}_{\mu\nu}$ associated with the metric obtained in the previous exercise."
> (Problem 2.2)
>
> "Write down the equations of motion for a free particle on a flat two-dimensional surface expressed in polar coordinates."
> (Problem 3.1)

Plane $x, y \rightarrow r, \varphi$ $(x^i \rightarrow x'^i)$.

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 = \delta_{ij}\, \mathrm{d}x^i\, \mathrm{d}x^j$$

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}, \qquad g'_{ij} = \frac{\partial x^k}{\partial x'^i} \frac{\partial x^l}{\partial x'^j} \delta_{kl}$$

Or differentiate $x = ..., y = ...$ directly:

$$\begin{cases} \mathrm{d}x = \mathrm{d}r\cos\varphi - r\,\mathrm{d}\varphi \sin\varphi \\ \mathrm{d}y = \mathrm{d}r \sin\varphi + r\,\mathrm{d}\varphi\cos\varphi \end{cases}$$

In this case we can even draw a general $\mathrm{d}s$ in polar coordinates, with radial component $\mathrm{d}r$ and a component $r\,\mathrm{d}\varphi$ perpendicular to it, the Pythagorean theorem gives $\mathrm{d}s^2 = \mathrm{d}r^2 + r^2 \mathrm{d}\varphi^2$ directly.

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\varphi^2$$

$$g_{rr} = 1, \quad g_{\varphi\varphi} = r^2, \quad g_{r\varphi} = g_{\varphi r} = 0, \quad g = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

Inverse metric:

$$g^{rr} = 1, \quad g^{\varphi\varphi} = \frac{1}{r^2}, \quad g^{r\varphi} = g^{\varphi r} = 0$$

We know that the geodesics will be straight line (the shortest path between two points is the straight line). Using polar coordinates we should be able to get some results familiar from Newtonian mechanics.

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu})$$

Any nonzero derivative has to be ∂_r. And it has to be the derivative of $g_{\varphi\varphi}$. The inverse metric doesn't interchange r, φ since it is diagonal. The only ones we have to check are $\Gamma^r_{\varphi\varphi}$ and $\Gamma^{\varphi}_{r\varphi} (= \Gamma^{\varphi}_{\varphi r})$. All the others vanish.

$$\Gamma^r_{\varphi\varphi} = -\frac{1}{2} \cdot 1 \cdot \partial_r r^2 = -r$$

$$\Gamma^{\varphi}_{r\varphi} = \frac{1}{2} \cdot \frac{1}{r^2} \cdot \partial_r r^2 = \frac{1}{r}$$

We can check this by writing down the geodesic equation, and see if we get things we can recognise.

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu_{\nu\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau} = 0$$

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} - r \frac{\mathrm{d}\varphi}{\mathrm{d}\tau} \frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = 0$$

$$\frac{\mathrm{d}^2 \varphi}{\mathrm{d}\tau^2} + 2 \cdot \frac{1}{r} \cdot \frac{\mathrm{d}r}{\mathrm{d}\tau} \cdot \frac{\mathrm{d}\varphi}{\mathrm{d}\tau}$$

Do you recognise these equations? You should! Let overdot be the derivative: $\dot{} = \frac{\mathrm{d}}{\mathrm{d}\tau}$

$$\begin{cases} \ddot{r} - r\,\dot{\varphi} = 0 \\ r\ddot{\varphi} + 2\,\dot{r}\dot{\varphi} = 0 \end{cases}$$

The first is the radial component of the acceleration, a_r. The second is the angular part of the acceleration, a_φ.

Note that the existence of $\Gamma^\lambda_{\mu\nu} \neq 0$ does not necessarily mean that the space is curved.

Exercises 2.4 and 3.2

> "Find the metric and affine connection on the surface of a two-sphere of radius a embedded in a Euclidean three-dimensional space."
> (Problem 2.4)
>
> "Find all geodesics on the surface of a two-sphere of radius a embedded in a Euclidean three-dimensional space."
> (Problem 3.2)

Two-dimensional sphere, S^2.

$$\begin{cases} x = a \sin\theta \cos\varphi \\ y = a \sin\theta \sin\varphi \\ z = a\cos\theta \end{cases}$$

$$\mathrm{d}s^2 = (a\,\mathrm{d}\theta)^2 + (a\sin\theta\,\mathrm{d}\varphi)^2 = a^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right)$$

$$g_{\theta\theta} = a^2, \quad g_{\varphi\varphi} = a^2 \sin^2\theta$$

$$g = a^2 \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}, \quad g^{-1} = a^{-2} \begin{pmatrix} 1 & 0 \\ 0 & \csc^2\theta \end{pmatrix}$$

The only non-vanishing components are $\Gamma^\theta_{\varphi\varphi}$ and $\Gamma^\varphi_{\theta\varphi} = \Gamma^\varphi_{\varphi\theta}$.

$$\Gamma^\theta_{\varphi\varphi} = -\ \sin\theta\cos\theta$$

$$\Gamma^\varphi_{\varphi\theta} = \frac{1}{2} \cdot \frac{1}{\sin^2\theta} \cdot 2\sin\theta\cos\theta = \cot\theta$$

Geodesic equations:

$$\begin{cases} \ddot{\theta} - \sin\theta\cos\theta\,\dot{\varphi}^2 = 0 \\ \ddot{\varphi} + 2\cot\theta \cdot \dot{\theta}\,\dot{\varphi} = 0 \end{cases}$$

Multiply the second equation by $\sin^2\theta$, and we get a total derivative.

$$\sin^2\theta\,\ddot{\varphi} + 2\sin\theta\cos\theta\cdot\dot{\theta}\dot{\varphi} = \frac{\mathrm{d}}{\mathrm{d}\tau}\big(\sin^2\theta\cdot\dot{\varphi}\big)$$

The solutions turn out to be great circles.

For $\theta = \frac{\pi}{2}$ the equations are $0=0$ and $\ddot{\varphi}=0$, and we get $\theta = \frac{\pi}{2}$, $\varphi = \varphi_0 + \omega\tau$. So instead of solving the equations generally, we can use adjust the coordinate system so that we get the apropriate initial conditions for this solution. Thus all the solutions are great circles, by symmetry.

An exercise not on the list.

Levi-Civita symbol $\varepsilon^{\mu\nu\kappa\lambda}$:

$$\varepsilon^{\mu\nu\kappa\lambda} = \begin{cases} 1 & \text{if } \mu,\nu,\kappa,\lambda \text{ is an even permutation of 0, 1, 2, 3} \\ -1 & \text{if } \mu,\nu,\kappa,\lambda \text{ is an odd permutation of } 0,1,2,3 \\ 0 & \text{otherwise (two or more indices equal)} \end{cases}$$

This holds in all coordinate systems.

Is this a tensor? In special relativity it is, but in general relativity it is not.

Transformation properties: $\varepsilon'^{\mu\nu\kappa\lambda} = \varepsilon^{\mu\nu\kappa\lambda}$ by definition.

Compare with

$$\underbrace{\frac{\partial x'^{\mu}}{\partial x^{\alpha}}\frac{\partial x'^{\nu}}{\partial x^{\beta}}\frac{\partial x'^{\kappa}}{\partial x^{\gamma}}\frac{\partial x'^{\lambda}}{\partial x^{\delta}}\varepsilon^{\alpha\beta\gamma\delta}}_{\text{completely antisymmetric} \Rightarrow \propto \varepsilon^{\mu\nu\kappa\lambda}}$$

["I'm extending the greek alphabet with letters from the beginning of the greek alphabet."]

What is the proportionality constant?

$$\frac{\partial x'^{\mu}}{\partial x^{\alpha}}\frac{\partial x'^{\nu}}{\partial x^{\beta}}\frac{\partial x'^{\kappa}}{\partial x^{\gamma}}\frac{\partial x'^{\lambda}}{\partial x^{\delta}}\varepsilon^{\alpha\beta\gamma\delta} = k\,\varepsilon^{\mu\nu\kappa\lambda}$$

Take $\mu\nu\kappa\lambda = 0123$.

$$k = \frac{\partial x'^{0}}{\partial x^{\alpha}}\frac{\partial x'^{1}}{\partial x^{\beta}}\frac{\partial x'^{2}}{\partial x^{\gamma}}\frac{\partial x'^{3}}{\partial x^{\delta}}\varepsilon^{\alpha\beta\gamma\delta} \equiv M^0_{\alpha}M^1_{\beta}M^2_{\gamma}M^3_{\delta}\varepsilon^{\alpha\beta\gamma\delta} \stackrel{\text{def}}{=\!=} \det M = \left|\frac{\partial x'}{\partial x}\right|$$

$$\varepsilon^{\mu\nu\kappa\lambda} = \varepsilon'^{\mu\nu\kappa\lambda} = \left|\frac{\partial x'}{\partial x}\right|^{1}\left(\frac{\partial x'^{\mu}}{\partial x^{\alpha}}\frac{\partial x'^{\nu}}{\partial x^{\beta}}\frac{\partial x'^{\kappa}}{\partial x^{\gamma}}\frac{\partial x'^{\lambda}}{\partial x^{\delta}}\varepsilon^{\alpha\beta\gamma\delta}\right)$$

$\varepsilon^{\mu\nu\kappa\lambda}$ is a *tensor density* of weight 1.

Important examples of densities:

- $|g|$ — what is the weight?
- d^4x (that we use when we integrate).