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Today we will essentialy work with the stuff we have defined so far, such as the metric tensor and the affine connection $\Gamma^\lambda_{\mu\nu}$, and we will talk about vectors and tensors in general relativity. We will see what the problem is with the affine connection, why it is not a tensor. We will be able to turn this to our advantage, when we define derivatives. But first a little something about what we had intended to do the last time.

We had talked about how particles move in the presence of gravity. We have seen the geodesic equation that governs particle motion; the very same equation without time, on some curved space, defines a geodesic — a sort of straight line on a curved surface (the only meaningful definition of "straight line" on a curved surface). We also talked about the Newtonian limit and found what exactly the Newtonian potential is.

"Time dilation" — a very physical effect of describing gravity with geometry. We had time dilation in special relativity, but this is really a different thing, and it is a pity that the same term is used for both things. What we will do is to consider a particle, or an observer, with some kind of internal clock, moving in some gravity field. (We will be more specific in a moment.) We have some coordinates, x, y, z, t. There is nothing that says that this is the ordinary Minkowski spacetime. The metric $g_{\mu\nu}(x)$ may vary from place to place. The particle traces out some world-line. The question we ask ourselves is, how often does the clock of the particle/the observer tick, as compared to the time coordinate t? What is $\mathrm{d}\tau$ (the observer's proper time) compared to $\mathrm{d}t$ (coordinate time)? The answer to this lies in special relativity, in the equivalence principle.

The equivalence principle states that there is a local inertial system, where $\mathrm{d}\tau = \sqrt{-\eta_{\alpha\beta}\,\mathrm{d}\xi^\alpha\,\mathrm{d}\xi^\beta}$ in accordance with special relativity. Changing coordinates to x, we have $\mathrm{d}\tau = \sqrt{-g_{\alpha\beta}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu}$. How that relates to $\mathrm{d}t$ is seen by dividing with $\mathrm{d}t$. (It is not a scalar, but that is fine here.)

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \sqrt{-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\frac{\mathrm{d}x^\nu}{\mathrm{d}t}}$$

In the end we want to compare different observers. Let us specialise on a "clock" at coordinate rest, $\frac{\mathrm{d}x^i}{\mathrm{d}t} = 0$.

$$\Rightarrow \frac{\mathrm{d}\tau}{\mathrm{d}t} = \sqrt{-g_{00}}$$

From yesterday we known that in the Newtonian limit $g_{00} = -1 - 2\,\Phi$, where Φ is the gravitational potential.

To be very concrete, what do we see? Say that the gravitational field is constant in time, such that the gravitational field from the earth, and study a clock (1) at sea level. Suppose furthermore that we have another observer (2) situated a bit higher up, and looking at the electromagnetic radiation from the clock (1), studying its ticking. Clock (1) ticks with a period time $\Delta\tau_1$, and the observer (2) measures a period time $\Delta\tau_2$. Both these have a relation to coordinate time.

$$\frac{\Delta\tau_1}{\Delta t} = \sqrt{-g_{00}}\Big|_1, \quad \frac{\Delta\tau_2}{\Delta t} = \sqrt{-g_{00}}\Big|_2$$

$$\frac{\Delta\tau_2}{\Delta\tau_1} = \sqrt{\frac{-g_{00}|_2}{-g_{00}|_1}} \simeq [\text{Newton}] \simeq \sqrt{\frac{1+2\Phi_2}{1+2\Phi_1}} \simeq \sqrt{1+2(\Phi_2-\Phi_1)} \approx 1+\Phi_2-\Phi_1 = 1 - MG\left(\frac{1}{r_2}-\frac{1}{r_1}\right)$$

If $r_2 > r_1$ we would expect to see a longer period: if light has to work against gravity it loses energy, and that lowers the frequency ($E = h\,\nu$). $\Delta\tau_2 > \Delta\tau_1$. This is *gravitational redshift*.

Vectors and tensors

$\mathrm{d}x^\mu$ is sort of the prototype vector. How does $\mathrm{d}x^\mu$ transform when we change coordinate system?

$$\mathrm{d}x'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}\mathrm{d}x^\nu$$

For an arbitrary vector, this is the rule:

$$A'^{\,\mu} = \frac{\partial x'^{\,\mu}}{\partial x^\nu} A^\nu$$

Tensors: $T^{\mu\nu}$ transform similarly for each index:

$$T'^{\,\mu\nu} = \frac{\partial x'^{\,\mu}}{\partial x^\rho} \frac{\partial x'^{\,\nu}}{\partial x^\sigma} T^{\rho\sigma}$$

Lower with $g_{\mu\nu}$:

$$A_\mu \equiv g_{\mu\nu} A^\nu$$

$$g_{\mu\nu} = \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} \eta_{\alpha\beta}$$

How does this transform? I'm not changing the ξ^α, I change the x^μ.

$$g'_{\mu\nu} = \frac{\partial \xi^\alpha}{\partial x'^\mu} \frac{\partial \xi^\beta}{\partial x'^{\,\nu}} \eta_{\alpha\beta} = \left[\frac{\partial \xi^\alpha}{\partial x'^{\,\mu}} = \frac{\partial \xi^\alpha}{\partial x^\rho} \frac{\partial x^\rho}{\partial x'^{\,\mu}}\right] = \frac{\partial x^\rho}{\partial x'^{\,\mu}} \frac{\partial x^\sigma}{\partial x'^{\,\nu}} g_{\rho\sigma}$$

If $\partial x'^{\,\mu}/\partial x^\nu$ is the matrix M, then $\partial x^\rho/\partial x'^\mu$ is the inverse matrix M^{-1}.

$$A'_\mu = [\text{“this is boring”}] = \frac{\partial x^\rho}{\partial x'^{\,\mu}} \frac{\partial x^\sigma}{\partial x'^{\,\nu}} g_{\rho\sigma} \frac{\partial x'^{\,\nu}}{\partial x^\lambda} A^\lambda$$

[“Once we've learned the rules, we can forget them.”]

$$\left[\frac{\partial x^\sigma}{\partial x'^{\,\nu}} \frac{\partial x'^{\,\nu}}{\partial x^\lambda} = \delta^\sigma_\lambda\right]$$

$$A'_\mu = \frac{\partial x^\rho}{\partial x'^{\,\mu}} g_{\rho\lambda} A^\lambda = \frac{\partial x^\rho}{\partial x'^{\,\mu}} A_\rho$$

$g_{\mu\nu}$ is a tensor!

$$A'_\mu B'^{\,\mu} = \ldots = A_\mu B^\mu$$

We can do scalar products!

We can multiply things together: $A_\mu B_\nu$ is a tensor.

There is one difficulty, and that's derivatives.

$$A'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} A_\nu$$

$$A'_\mu = \frac{\partial x^\nu}{\partial x'^{\,\mu}} A_\nu$$

$\{\mathrm{d}x^\mu\}$ basis, $A = \mathrm{d}x^\mu A_\mu$. We call A the “1-form”.

$\left\{\frac{\partial}{\partial x^\mu}\right\}$ basis, $B = B^\mu \frac{\partial}{\partial x^\mu}$. This is a “vector field”.

(The two terms above are from differential geometry, and will not figure prominently in this course.)

Let us take a close look at the connection. The affine connection is

$$\Gamma^{\lambda}_{\mu\nu} = \frac{\partial x^{\lambda}}{\partial \xi^{\alpha}} \frac{\partial^2 \xi^{\alpha}}{\partial x^{\mu} \partial x^{\nu}}$$

This was one possible definition. We can also define it in terms of derivatives of the metric:

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}(\partial_{\mu} g_{\nu\rho} + \partial_{\nu} g_{\mu\rho} - \partial_{\rho} g_{\mu\nu})$$

We will use the first definition now.

$$\Gamma'^{\lambda}_{\mu\nu} = \frac{\partial x'^{\lambda}}{\partial \xi^{\alpha}} \frac{\partial^2 \xi^{\alpha}}{\partial x'^{\mu} \partial x'^{\nu}} = \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\rho}}{\partial \xi^{\alpha}} \cdot \frac{\partial}{\partial x'^{\mu}} \left(\frac{\partial \xi^{\alpha}}{\partial x^{\sigma}} \cdot \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \right)$$

$$\Gamma'^{\lambda}_{\mu\nu} = \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \frac{\partial x^{\tau}}{\partial x'^{\mu}} \Gamma^{\rho}_{\sigma\tau} + \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial^2 x^{\rho}}{\partial x'^{\mu} \partial x'^{\nu}}$$

Γ is not a tensor.

Derivatives

On scalar: $\partial_{\mu}\phi$:

$$(\partial_{\mu}\phi)' = \frac{\partial \phi}{\partial x'^{\mu}} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} \frac{\partial \phi}{\partial x^{\nu}} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} \partial_{\nu}\phi$$

This was a vector.

On a vector V^{μ}

$$(\partial_{\nu} V^{\mu})' = \frac{\partial V'^{\mu}}{\partial x'^{\nu}} = \frac{\partial x^{\rho}}{\partial x'^{\mu}} \frac{\partial}{\partial x^{\rho}} \left(\frac{\partial x'^{\mu}}{\partial x^{\sigma}} V^{\sigma} \right)$$

When the derivative hits the vector, we get desirable things. But when it hits the transformation matrix we get bad stuff.

$$(\partial_{\nu} V^{\mu})' = \frac{\partial x^{\rho}}{\partial x'^{\nu}} \frac{\partial x'^{\mu}}{\partial x^{\sigma}} \partial_{\rho} V^{\sigma} + \frac{\partial x^{\rho}}{\partial x'^{\nu}} \frac{\partial^2 x'^{\mu}}{\partial x^{\rho} \partial x^{\sigma}} V^{\sigma}$$

The plain derivative of a vector, is not a vector. The second term above spoils everything.

V^{μ} a vector $\not\Rightarrow \partial_{\mu} V^{\nu}$ a vector!

∂_{μ} is not a covariant operator. (A covariant operator takes vectors and tensors, to vectors and tensors.)

- Idea: Calculate $\frac{\partial A^{\alpha}}{\partial \xi^{\beta}}$ and transform.

The failure of the derivative to be a tensor, looks very much like the failure of the $\Gamma^{\lambda}_{\mu\nu}$ to be a tensor. The idea is to combine them into something that will transform the right way:

$$\mathrm{D}_{\mu} V^{\nu} \equiv \partial_{\mu} V^{\nu} + \Gamma^{\nu}_{\mu\lambda} V^{\lambda}$$

This is the only way of defining a covariant derivative.

$$\partial_{\mu} W_{\nu} - \Gamma^{\lambda}_{\mu\nu} W_{\lambda}$$

The signs here are not obvious. You have to check it. But these two latter expressions *are* tensors.

Particle:

$$\frac{\mathrm{d}^2\xi^\alpha}{\mathrm{d}\tau^2} = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau} = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\cancel{\partial}}{\cancel{\partial} x^\mu}, \quad \frac{\mathrm{D}}{\mathrm{D}\tau} = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\mathrm{D}_\mu$$

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \text{ is clearly a vector}$$

$$\frac{\mathrm{D}}{\mathrm{D}\tau}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = 0$$

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu_{\nu\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau}$$

Check, as a homework, that $\mathrm{D}_\mu g_{\nu\lambda} = 0$. $g_{\mu\nu}$ is covariantly constant (which is a very nice property).