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Today we will actually start describing gravity. When doing that we gradually introduce the relevant mathematics involved, the mathematics needed to describe curved space and curved spacetime. The level of mathematical rigour will be very low, we will do what we actually need. We will normally not care at all about what spaces the functions belong to, how many times they are differentiable and so on.

We have mentioned the need of a field theory of gravity. What happens if we move the sun? Locality is an important principle, normally, in physics. The situation is analogous in "any reasonable theory". No action at a distance. While we may talk about a gravitational potential in Newtonian mechanics, we don't know how it behaves and what happens to it when things move around in it.

Gravity is a sort of a funny force, because you don't feel it if you are in free fall. While you are falling, it is like being in outer space: you don't feel gravity. This leads us to the equivalence principle:

Equivalence principle: There is no measurable difference between gravity and the effect of acceleration.

This means that [draws an elevator]. It is broken, standing on the ground. No windows. This is a thought experiment, in the same vein as those imagined by Galileo Galilei. You can drop an apple in the elevator, and see how it falls.

Next imagine the elevator in outer space, accelerating upward with an acceleration g. The inhabitants of the elevator experience an inertial force. If you drop an apple in the elevator, it continue to travel at its initial velocity, while the elevator accelerates away from it. So the apple seems, relative to the elevator, to be accelerating downward with the same acceleration g.

Let us now imagine that these two scenarios are the same. (That there is no measurable difference.) There is one assumption here, that is "sort of funny". Consider two different kinds of apple, made of different kind of materials. In the free space scenario they both remain at the same velocity, and the inhabitants of the elevator measure them falling with the same acceleration g. In the gravitational case, one might imagine that the fall with different accelerations. It depends on the nature of gravity. Experiments have been done, e.g. by the Hungarian physicist Eötvös.

Thinking for the moment in Newtonian terms, $m_\mathrm{i}\, a = m_\mathrm{g}\, g$, where m_i is the *inertial* mass. m_g is the gravitational mass. There is really no reason that they ought to be the same: here we need experimental input. The experiments say that $m_\mathrm{i} = m_\mathrm{g}$. This is an assumption, and we will now assume this.

From now on we will describe gravity in terms of acceleration.

This is a bit absurd. Thinking about a small elevator is one thing, but consider the entire earth! It would make sense to say that this room is accelerating upwards, but the earth is round and they teach physics in Australia too. If all rooms are accelerating upwards, it would seem that the circumference of the earth itself would accelerate, that the earth would expand, accelerating. That is not the case. We have to consider small elevators for the equivalence principle to hold. The equivalence principle holds locally.

How do we use this equivalence principle? We reformulate the statement a bit:

• There is always a set of (local) inertial systems where physics look like special relativity. The freely falling system will be an inertial frame in the sense of special relativity.

What was really the hurdle at this point is that, at this point, we need to do geometry. Consider spatial geometry, like a plane, where the Pythagorean theorem holds. And consider curved surfaces, such as the surface of a sphere. Given any point at a curved space (not necessarily a sphere), if you don't go too far from this point the surface looks sort of flat. You can always approximate the space close to a given space, with a tangent plane in that point. This is the reason why people once thought the earth was approximately flat (give or take a few mountains and valleys and stuff).

• In space, there is always a (local) tangent plane.

We can do something similar in curved spacetime, with an inertial frame being a "tangent plane" at an event (a point in spacetime), a tangent Minkowski space. This was the brave leap that Einstein did. Now, we just have to formulate the mathematics of curved spacetime. We know what happens in a local Minkowski space, and then we just do a coordinate transformation. We will, rather soon, get rid of the inertial coordinates altogether, and work in curved spacetime.

Consider a particle moving under influence of gravity:

What do we know? In general we don't know anything, but we know that there is an inertial frame where the effects of gravity are not felt. (Well, there are several, related by Lorentz transformations.) In the (local) inertial system with coordinates ξ^α, the velocity is constant:

$$\frac{\mathrm{d}^2\,\xi^\alpha}{\mathrm{d}\tau^2} = 0, \quad \text{where } \mathrm{d}\tau^2 = -\,\eta_{\alpha\beta}\,\mathrm{d}\xi^\alpha\,\mathrm{d}\xi^\beta.$$

Now we change coordinates to some x^μ on spacetime. We can see x^μ as functions of the ξ^α, or the ξ^α as functions of x^μ. Using the chain rule of differentiation:

$$0 = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial\xi^\alpha}{\partial x^\mu}\cdot\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\right) = \frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2} + \frac{\partial^2\xi^\alpha}{\partial x^\mu\,\partial x^\nu}\cdot\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\cdot\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \frac{\partial\xi^\alpha}{\partial x^\mu}\left[\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2} + \frac{\partial x^\mu}{\partial\xi^\beta}\cdot\frac{\partial^2\xi^\beta}{\partial x^\mu\,\partial x^\nu}\,\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\,\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right]$$

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2} + \underbrace{\frac{\partial x^\mu}{\partial\xi^\beta}\cdot\frac{\partial^2\xi^\beta}{\partial x^\nu\,\partial x^\lambda}}_{=\Gamma^\mu_{\nu\lambda}}\,\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\,\frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau} = 0$$

$\Gamma^\mu_{\nu\lambda}$ is called the *affine connection*. In some sense $\Gamma^\mu_{\nu\lambda}$ encodes the information about gravity. It is a sign of gravity — or a sign of a silly choice of coordinates. (Think about a plane, τ being length along the path. $\frac{\mathrm{d}^2\,\xi^\alpha}{\mathrm{d}\tau^2} = 0$ gives a straight line. If we choose polar coordinates x^μ, we will get a nonzero $\Gamma^\mu_{\nu\lambda}$, so that in itself does not imply gravity.)

If gravity is really present (in the sense that the spacetime is *really* curved), then $\Gamma^\mu_{\nu\lambda}$ (which looks like a tensor, but is not) then $\Gamma^\mu_{\nu\lambda} \neq 0$. But it can be nonzero on flat space if we use non-inertial coordinates. A tensor is either zero, or it is not, it cannot choose to be zero in one set of coordinates without being it in the others as well.

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu_{\nu\lambda}\,\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\,\frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau} = 0$$

is called the *geodesic equation*. The solution of this equation, the world-line of a particle, is called a *geodesic*. (Swedish: *geodet*.) As an exercise, solve the geodesic equation for a sphere, and see that the geodesics are the great circles.

$$\mathrm{d}s^2 = -\,\mathrm{d}\tau^2 = \eta_{\alpha\beta}\,\mathrm{d}\xi^\alpha\mathrm{d}\xi^\beta = \underbrace{\eta_{\alpha\beta}\,\frac{\partial\xi^\alpha}{\mathrm{d}x^\mu}\,\frac{\partial\xi^\beta}{\partial x^\nu}}_{=g_{\mu\nu}}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$$

Formally, it looks like special relativity, $\mathrm{d}s^2 = \eta_{\alpha\beta}\,\mathrm{d}\xi^\alpha\mathrm{d}\xi^\beta = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$, but $g_{\mu\nu}$ can be more general things not considered in special relativity. $g_{\mu\nu}$ is called the *metric tensor*, a 4×4 symmetric matrix (10 independent components). It is a function of the coordinates x^μ, $g_{\mu\nu}(x)$. $\Gamma^\mu_{\nu\lambda}$ encodes how $g_{\mu\nu}$ changes when we move around.

Now, let us get rid of ξ^α, let us get rid of the tangent Minkowski space. We base our theory on $g_{\mu\nu}$ directly, rather than inertial coordinates ξ^α. $\Gamma^\mu_{\nu\lambda}$ looks like it is going to make trouble, but we can get around that.

$$g_{\mu\nu} = \frac{\partial\xi^\alpha}{\partial x^\mu}\,\frac{\partial\xi^\beta}{\partial x^\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$$

Using the notation $\partial_\lambda = \frac{\partial}{\partial x^\lambda}$, and taking the derivative:

$$\partial_\lambda g_{\mu\nu} = \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\lambda} \frac{\partial \xi^\beta}{\partial x^\nu} \eta_{\alpha\beta} + \frac{\partial^2 \xi^\alpha}{\partial x^\nu \partial x^\lambda} \frac{\partial \xi^\beta}{\partial x^\mu} \eta_{\alpha\beta}$$

This looks a bit like $\Gamma^\mu_{\nu\lambda}$, but not quite.

$$\frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu} = \frac{\partial \xi^\alpha}{\partial x^\lambda} \Gamma^\lambda_{\mu\nu}$$

$$\partial_\lambda g_{\mu\nu} = \Gamma^\rho_{\mu\lambda} \underbrace{\frac{\partial \xi^\alpha}{\partial x^\rho} \frac{\partial \xi^\beta}{\partial x^\nu} \eta_{\alpha\beta}}_{= g_{\rho\nu}} + \Gamma^\rho_{\nu\lambda} \underbrace{\frac{\partial \xi^\alpha}{\partial x^\rho} \frac{\partial \xi^\beta}{\partial x^\mu} \eta_{\alpha\beta}}_{= g_{\rho\mu}}$$

$$\partial_\lambda g_{\mu\nu} = \Gamma^\rho_{\mu\lambda} g_{\rho\nu} + \Gamma^\rho_{\nu\lambda} g_{\rho\mu}$$

$$\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu} = [\text{homework}] = 2\Gamma^\rho_{\mu\nu} g_{\rho\lambda}$$

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu})$$

The inverse of $g_{\mu\nu}$ is $(g^{-1})^{\mu\nu} = g^{\mu\nu}$. This is just notation. It is the ordinary matrix inverse.

Now we have $\Gamma^\lambda_{\mu\nu}$ in terms of the metric tensor $g_{\mu\nu}$, and we do not have to resort to inertial systems, no ξ^α needed.

Now, can we reproduce Newtonian gravity in this theory?

The Newtonian limit:

We assume (1) that velocity $\ll 1$.

$$\frac{\mathrm{d}x^i}{\mathrm{d}\tau} \ll \frac{\mathrm{d}t}{\mathrm{d}\tau}, \quad t \equiv x^0$$

We also assume (2) that $\partial_0 g_{\mu\nu} = 0$, since we can only handle static situations using Newtonian gravity.

$$0 \simeq \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu_{00} \left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2$$

$$\Gamma^\mu_{00} = -\frac{1}{2} g^{\mu\nu} \partial_\nu g_{00}$$

Newtonian gravity only applies to weak gravitational fields (no black holes here), so we assume (3) that $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$.

$$\Gamma^\mu_{00} = -\frac{1}{2} \eta^{\mu\nu} \partial_\nu h_{00}$$

$$0 \simeq \frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} - \frac{1}{2} \partial^i h_{00}$$

$$\frac{\mathrm{d}^2 \boldsymbol{x}}{\mathrm{d}t^2} = \frac{1}{2} \nabla h_{00}$$

By setting $h_{00} = -2\Phi$,

$$\frac{\mathrm{d}^2 \boldsymbol{x}}{\mathrm{d}t^2} = -\nabla \Phi.$$

Only one component of the entire metric becomes relevant in this limit. Φ is the gravitational potential of Newtonian gravity.