2008–10–30

1. Proper time $\mathrm{d}\tau$ is defined by

$$\mathrm{d}\tau^2 \equiv \mathrm{d}t^2 - \mathrm{d}\boldsymbol{x}^2 = -\,\eta_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta$$

where we are working in units where $c=1$. $\alpha=(0,i)$. We are using the metric

$$\eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

A Lorentz transformation is a coordinate transformation $x^\alpha \overset{\mathrm{LT}}{\mapsto} x'^{\,\alpha} = \Lambda^\alpha_{\ \beta}\, x^\beta + a^\alpha$, where $\Lambda^\alpha_{\ \beta}$ obeys $\eta_{\alpha\beta} = \Lambda^\gamma_{\ \alpha}\Lambda^\delta_{\ \beta}\,\eta_{\gamma\delta}$. For differentials $\mathrm{d}x^\alpha$ we have $\mathrm{d}x^\alpha \overset{\mathrm{LT}}{\mapsto} \mathrm{d}x'^\alpha = \Lambda^\alpha_{\ \beta}\,\mathrm{d}x^\beta$.

$$\Rightarrow \mathrm{d}\tau^2 \overset{\mathrm{LT}}{\mapsto} \mathrm{d}\tau'^{\,2} = -\,\eta_{\alpha\beta}\,\mathrm{d}x'^{\,\alpha}\,\mathrm{d}x'^{\,\beta} = -\,\eta_{\alpha\beta}\,\Lambda^\alpha_{\ \gamma}\,\Lambda^\beta_{\ \delta}\,\mathrm{d}x^\gamma\,\mathrm{d}x^\delta = -\,\eta_{\gamma\delta}\,\mathrm{d}x^\gamma\,\mathrm{d}x^\delta \equiv \mathrm{d}\tau^2$$

$\mathrm{d}\tau'^{\,2} = \mathrm{d}\tau^2 \Rightarrow \mathrm{d}\tau$ is invariant under Lorentz transformations.

<u>Note:</u> For 4-vectors $x^\alpha = \eta^{\alpha\beta} x_\beta \Rightarrow$

$$\alpha = 0: \quad x^0 = \eta^{00} x_0 = -\,x_0$$

$$\alpha = i: \quad x^i = \eta^{i\beta} x_\beta = \eta^{ii} x_i\,[\text{no sum over } i] = x_i$$

2) Standard Lorentz boost

$$\begin{cases} t' = \gamma(v)\cdot(t - v\,x) \\ x' = \gamma(v)\cdot(x - v\,t) \\ y' = y \\ z' = z \end{cases}$$

We can think of x^α as a column vector and $\Lambda^\alpha_{\ \beta}$ as a matrix, with α enumerating the rows, β the columns.

$$x^\alpha = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \quad \overset{\mathrm{LT}}{\mapsto} \quad x'^{\,\alpha} = \begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \Lambda^\alpha_{\ \beta} x^\beta = \begin{pmatrix} \gamma & -\,v\,\gamma & 0 & 0 \\ -\,v\,\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

The Lorentz factor γ is defined

$$\gamma \equiv \frac{\mathrm{d}t}{\mathrm{d}\tau} = \left(\frac{\mathrm{d}\tau^2}{\mathrm{d}t^2}\right)^{-1/2} = \left(1 - \sum_i \left(\frac{\mathrm{d}x^i}{\mathrm{d}t}\right)^2\right)^{-1/2} = \left(1 - \boldsymbol{v}^2\right)^{-1/2} = \frac{1}{\sqrt{1 - \boldsymbol{v}^2}}$$

For arbitrary $\boldsymbol{v} = v^i$:

$$\begin{cases} \Lambda^0_{\ 0} = \gamma \\ \Lambda^0_{\ i} = -\,v_i\gamma \\ \Lambda^i_{\ 0} = -\,v_i\gamma \\ \Lambda^i_{\ j} = \delta^i_j + v^i\,v_j\,\frac{(\gamma - 1)}{\boldsymbol{v}^2} \end{cases}$$

The Lorentz transformation condition $\eta_{\alpha\beta} = \Lambda^\gamma_{\ \alpha}\,\Lambda^\delta_{\ \beta}\,\eta_{\gamma\delta}$.

$\alpha = \beta = 0$:

$$\Lambda^{\gamma}{}_{0}\,\Lambda^{\delta}{}_{0}\,\eta_{\gamma\delta} = \Lambda^{0}{}_{0}\,\Lambda^{\delta}{}_{0}\eta_{0\delta} + \Lambda^{i}{}_{0}\,\Lambda^{\delta}{}_{0}\,\eta_{i\delta} = \Lambda^{0}{}_{0}\,\Lambda^{0}{}_{0} + \sum_{i}\,\Lambda^{i}{}_{0}\,\Lambda^{i}{}_{0}\,\eta_{ii} = \gamma^2(-1) + \sum_{i}\,(-v_i\gamma)^2(+1) =$$

$$= -\gamma^2\big(1 - v^2\big) = -1 = \eta_{00}$$

$\alpha = 0, \beta = i$.

$$\Lambda^{\gamma}{}_{0}\,\Lambda^{\delta}{}_{i}\,\eta_{0\gamma\delta} = \Lambda^{0}{}_{0}\,\Lambda^{\delta}{}_{i}\eta_{0\delta} + \Lambda^{j}{}_{0}\Lambda^{\delta}{}_{i}\eta_{j\delta} = \Lambda^{0}{}_{0}\,\Lambda^{0}{}_{i}\eta_{00} + \sum_{j}\,\Lambda^{j}{}_{0}\,\Lambda^{j}{}_{i}\eta_{jj} =$$

$$= \gamma(-v_i\gamma)(-1) + \sum_{j}\,(-v_j\,\gamma)\bigg(\delta_{ij} + v_i v_j\,\frac{(\gamma-1)}{\boldsymbol{v}^2}\bigg) =$$

$$= \gamma^2\,v_i - v_i\gamma - \underbrace{\sum_{j}\,(v_j)^2\cdot\frac{1}{\boldsymbol{v}^2}}_{=1}\,v_i\,(\gamma-1)\,\gamma = v_i\,\big(\gamma^2 - \gamma - \gamma^2 + \gamma\big) = 0 = \eta_{0i}$$

The metric is symmetric $\Rightarrow \Lambda^{\gamma}{}_{i}\Lambda^{\delta}{}_{0}\eta_{\gamma\delta} = 0 = \eta_{i0}$.

$\alpha = i, \beta = j$: $\Lambda^{\gamma}{}_{i}\,\Lambda^{\delta}{}_{j}\,\eta_{\gamma\delta} = \ldots = \delta_{ij} \Rightarrow$ The transformation condition $\eta_{\alpha\beta} = \Lambda^{\gamma}{}_{\alpha}\,\Lambda^{\delta}{}_{\beta}\,\eta_{\gamma\delta}$.

$$\Lambda_{\alpha\beta} = \eta_{\alpha\delta}\Lambda^{\delta}{}_{\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \gamma & v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

3) "Suppose that a particle is moving with velocity u at an angle θ from the x-axis in the xy-plane of a system S. What is the corresponding angle θ' in a system S' moving with velocity v in the x-direction relative to S?"

Let U^{α} be the 4-velocity of the particle

$$U^{\alpha} \equiv \frac{\mathrm{d}x^{\alpha}}{\mathrm{d}\tau} = \frac{\mathrm{d}t}{\mathrm{d}\tau}\,\frac{\mathrm{d}x^{\alpha}}{\mathrm{d}t} = \gamma\cdot\frac{\mathrm{d}x^{\alpha}}{\mathrm{d}t} = \gamma\,(1, v_x, v_y, v_z)$$

Write down U^{α} annd U'^{α}:

$$U^{\alpha} = \gamma(u)\cdot(1, u\cos\theta, u\sin\theta, 0) \text{ in } S$$

$$U'^{\alpha} = \gamma(u')\cdot(1, u'\cos\theta', u'\sin\theta', 0) \text{ in } S'$$

Since U^{a} is a 4-vector, these are related thorugh the Lorentz transformation connecting S and S'.

$$U'^{\alpha} = \Lambda^{\alpha}{}_{\beta}U^{\beta} \text{ where } \Lambda^{\alpha}{}_{\beta} = \begin{pmatrix} \gamma(v) & -v\,\gamma(v) & 0 & 0 \\ -v\,\gamma(v) & \gamma(v) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\alpha = 0$:

$$\gamma(u') = U'^{0} = \Lambda^{0}{}_{\beta}U^{\beta} = \Lambda^{0}{}_{0}\,U^0 + \Lambda^{0}{}_{1}U^1 = \gamma(v)\,\gamma(u) + (-v\,\gamma(v))\,\gamma(u)\,u\cos\theta$$

Rather useless for determining θ', since θ' does not enter into this expression.

$\alpha = 1$:

$$\gamma(u')\, u' \cos\theta' = U'^1 = \Lambda^1{}_\beta U^\beta = \Lambda^1{}_0 U^0 + \Lambda^1{}_1 U^1 = (-v\,\gamma(v))\,\gamma(u) + \gamma(v)\,u\cos\theta\,\gamma(u) =$$

$$= \gamma(v)\,\gamma(u)\,(u\cos\theta - v) \tag{1}$$

$\alpha = 2$:

$$\gamma(u')\, u'\, \sin\theta' = U'^2 = \Lambda^2{}_\beta U^\beta = \Lambda^2{}_2 U^2 = \gamma(u)\, u \sin\theta \tag{2}$$

Divide (1) by (2) to determine θ':

$$\Rightarrow \tan\theta' = \frac{u\sin\theta}{\gamma(v)(u\cos\theta - v)}$$

4. "We have two rockets, A and B, moving in some system S with velocities u and v, respectively. Findd therelative velocity of B in A's reference frame!"

Let U^α be the 4-velocity of A, and V^α be the 4-velocity of B.

Form the Lorentz scalar (Lorentz invariant):

$$U^\alpha V_\alpha = \eta_{\alpha\beta} U^\alpha V^\beta$$

This has the same value in any frame $\Rightarrow$ we can choose any frame!

We choose the rest frame of A: $U^\alpha = (1, \mathbf{0})$. In this frame the 4-veolocity of B is $V^\alpha = \gamma(\boldsymbol{v}_{\text{rel}})(1, \boldsymbol{v}_{\text{rel}})$

$$\Rightarrow U^\alpha V_\alpha = \eta_{\alpha\beta} U^\alpha U^\beta = \eta_{00} U^0 V^0 = (-1)\cdot 1 \cdot \gamma(\boldsymbol{v}_{\text{rel}}) = -\frac{1}{\left(1 - \boldsymbol{v}_{\text{rel}}^2\right)^{1/2}}$$

$$\Rightarrow (U^\alpha V_\alpha)^2 = \frac{1}{1 - \boldsymbol{v}_{\text{rel}}^2} \Rightarrow |\boldsymbol{v}_{\text{rel}}| = \left(1 - \frac{1}{(U^\alpha V_\alpha)^2}\right)^{1/2}$$

Measure the veolocities U^α, V^α in *any* frame to get $U^\alpha V_\alpha \rightsquigarrow |\boldsymbol{v}_{\text{rel}}|$.

5. "Show that

$$\begin{cases} \dfrac{\partial}{\partial x^\alpha} F^{\alpha\beta} = -J^\beta \\ \varepsilon^{\alpha\beta\gamma\delta} \dfrac{\partial}{\partial x^\beta} F_{\gamma\delta} = 0 \end{cases}$$

are the usual Maxwell equations."

The electromagnetic field strength tensor $F^{\alpha\beta} = -F^{\beta\alpha}$, $F^{0i} = E^i$, $F^{ij} = \varepsilon^{ijk} B_k$

$$F^{\alpha\beta} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix}$$

The first equation:

$$\partial_\alpha F^{\alpha\beta} = -J^\beta$$

$\beta = 0$:

$$\partial_0 F^{00} + \partial_i F^{i0} = -J^0 \quad \Rightarrow \quad \partial_0(0) + \partial_i(-E^i) = -\varepsilon \quad \Rightarrow \quad \partial_i E^i = \varepsilon \quad \Rightarrow \quad \boxed{\nabla \cdot \boldsymbol{E} = \varepsilon}$$

$\beta = j$:

$$\partial_0 \underbrace{F^{0j}}_{=E^j} + \partial_i \underbrace{F^{ij}}_{=\varepsilon^{ijk} B_k} = -J^j \quad \Rightarrow \partial_0 E^i - \varepsilon^{ijk} \partial_i B_k = -j^j$$

$$\Rightarrow \boxed{\nabla \times \boldsymbol{B} = \boldsymbol{j} + \frac{\partial \boldsymbol{E}}{\partial t}}$$

The second equation:

$$\varepsilon^{\alpha\beta\gamma\delta} \partial_\beta F_{\gamma\delta} = 0$$

$\alpha = 0$:

$$0 = \varepsilon^{0ijk} \partial_i F_{jk} = \varepsilon^{0ijk} \partial_i (\varepsilon_{jkl} B_l) = \varepsilon^{0ijk} \varepsilon_{jkl} \partial_i B_l = 2\, \delta^i_l\, \partial_i B_l \quad \Rightarrow \partial_i B^i = 0$$

$$\boxed{\nabla \cdot \boldsymbol{B} = 0}$$

$\alpha = i$:

$$0 = \varepsilon^{ijk0} \partial_j F_{k0} + \varepsilon^{ij0k} \partial_j F_{0k} + \varepsilon^{i0jk} \partial_0 F_{jk} = 2\, \varepsilon^{ij0k} \partial_j \underbrace{F_{0k}}_{-E_k} + \varepsilon^{i0jk} \partial_0 \underbrace{F_{jk}}_{\varepsilon_{jkl} B_l} =$$

$$= -2 \varepsilon^{ij0k} \partial_j E_k - \underbrace{\varepsilon^{0ijk} \varepsilon_{jkl}}_{=2\delta^i_l} \partial_0 B_l = -2\, \varepsilon^{0ijk} \partial_j E_k - 2\, \delta_0 B_i$$

$$\Rightarrow \quad \boxed{\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}}$$