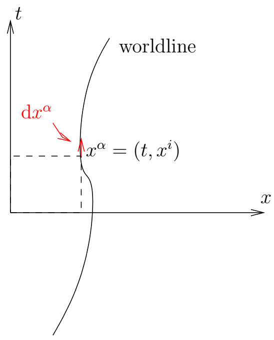
2008–10–28

Today: particle kinematics in special relativity, something about charges, currents, and we will begin to get into the discussion about charges and currents for gravity, and see what that means.

Particle moving in Minkowski spacetime.



The velocity presents a problem. The Newtonian version, $\boldsymbol{v} = \dot{\boldsymbol{x}} = \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}$, does not work. It is not appropriate to write $v^\alpha = \frac{\mathrm{d}x^\alpha}{\mathrm{d}t}$, not because it does not exist, but it is not a vector! We differentiate with respect to one of the coordinates. We need to differentiate with respect to a *scalar*.

$$\eta = \mathrm{diag}(-1, 1, 1, 1)$$

$$\mathrm{d}s^2 = \eta_{\alpha\beta}\, \mathrm{d}x^\alpha\, \mathrm{d}x^\beta < 0$$

$\mathrm{d}x^\alpha$ is time-like. $\mathrm{d}\tau = \sqrt{-\mathrm{d}s^2}$: τ is the proper time of the particle. This is a scalar. So, we can differentiate with respect to τ, and get back a vector:

$$v^\alpha = \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}$$

Using vectors enables us to write equations in a way that does not depend on the choice of coordinate system.

Now, in components. Three dimensional velocity v^i is defined as $v^i = \frac{\mathrm{d}x^i}{\mathrm{d}t}$, so:

$$\mathrm{d}x^\alpha = \left(\mathrm{d}t, \mathrm{d}x^i\right) = (\mathrm{d}t, v^i\, \mathrm{d}t)$$

$$\mathrm{d}s^2 = \mathrm{d}t^2\left(-1 + v^2\right)$$

$$\mathrm{d}\tau = \mathrm{d}t\, \sqrt{1 - v^2} = \frac{\mathrm{d}t}{\gamma(v)}$$

(This is the formula for time dilation.)

$$v^\alpha = \gamma(v)\, \frac{\mathrm{d}x^\alpha}{\mathrm{d}t} = \gamma(v)\, \left(1, v^i\right)$$

Acceleration four-vector:

$$a^\alpha = \frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\tau^2}$$

$$V^\alpha = \frac{\mathrm{d}x^\alpha}{\sqrt{-\mathrm{d}x_\alpha \mathrm{d}x^\alpha}} \quad \Rightarrow V^2 = -1$$

V is a normalised tangent vector to the world line.

4-momentum. In three dimensions we have $\boldsymbol{p} = m\,\boldsymbol{v}$, so we take

$$P^\alpha = m\,V^\alpha = m\,\gamma(v)\left(1, v^i\right) = \left(p^0, p^i\right)$$

$$p^0 = m\,\gamma(v) = E; \quad p^i = m\,\gamma(v)\,v^i$$

Some times people refer to $m\,\gamma(v)$ as relativistic mass or simply mass, and then m would be the scalar mass. We will always refer to m as the mass. We call $m\,\gamma(v)$ the energy. (We can make a Taylor expansion of the energy: $E = m + \frac{1}{2}m\,v^2 + \mathcal{O}(v^4)$).

Current and charges in a relativistic context. (Much of it is just common sense, but it becomes rather nice in the relativistic setting.)

[The minus sign in the definition of the Lagrangian in classical mechanics, $L = T - V$, is somehow related to the minus sign in the metric. You don't need to remember this.]

Take a charge distribution (charge density) $\rho(\boldsymbol{r}, t)$. If we take a small element of this, that element will have some well defined velocity $\boldsymbol{v}(\boldsymbol{r}, t)$. Then we can define a current density, $\boldsymbol{j} = \rho\,\boldsymbol{v}$.

Given ρ and $\boldsymbol{j}$ (and conservation of charge), we have the equation of continuity.

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\boldsymbol{j} = 0$$

(Consider a fixed volume V, any volume, and take the charge $Q = \int_V \rho\,\mathrm{d}V$.

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \begin{cases} \displaystyle\int_V \frac{\partial\rho}{\partial t}\,\mathrm{d}V \\ \displaystyle -\int_{S=\partial V} \boldsymbol{j}\cdot\mathrm{d}\boldsymbol{S} = -\int_V \nabla\cdot\boldsymbol{j}\,\mathrm{d}V \end{cases}$$
)

I would like to write down, not a ρ and a $\boldsymbol{j}$, I want something four-dimensional. One nice thing would be if the ρ was actually the fourth component of the current. And... it turns out it is.

With ρ_0 being the charge density in the local rest system, we have $\rho = \gamma(v)\,\rho_0$.

$$\rho_0\,V^\alpha = J^\alpha$$

$$J^\alpha = \rho_0\,\gamma(v)\left(1, v^i\right) = \left(\rho, j^i\right)$$

$$\partial_\alpha = \frac{\partial}{\partial x^\alpha} = \left(\frac{\partial}{\partial t}, \nabla\right)$$

$$\frac{\partial}{\partial x^\alpha}\,x^\beta = \delta^\beta_{\ \alpha} \quad \text{(unit matrix)}$$

We cannot write $\delta^{\cancel{\alpha}\beta}$, it is not a tensor.

$$\partial_\alpha J^\alpha = \frac{\partial\rho}{\partial t} + \nabla\cdot\boldsymbol{j} = 0$$

Now comes the difficult part; some reasoning about electromagnetism and gravity.

Electromagnetism	Gravity
charge	4-momentum (it does not have to have a rest mass, light also gravitates)
4-current	stress-energy tensor, or energy-momentum tensor

The 4-current is the right hand side of Maxwell's equations, so the stress-energy tensor then ought to be the right hand side of... whose equations? Einstein's equations.

Take a mass distribution, with energy density ρ, with each volume element moving in some direction with velocity $\boldsymbol{v}$. The quantity that is conserved, is the 4-momentum.

Energy density: $\rho = (\gamma(v))^2 \rho_0$.

Momentum density: $\rho\, \boldsymbol{v} = \gamma^2(v)\, \rho_0 \boldsymbol{v}$. $(P^\alpha = m\, \gamma(v)\, (1, v^i))$.

Energy flow: $\rho\, \boldsymbol{v} = \gamma^2 \rho_0\, \boldsymbol{v}$.

Momentum flow: $\rho\, v^i v^j = \gamma^2(v)\, \rho_0\, v^i\, v^j$.

$$T^{\alpha\beta} = \left(\begin{array}{c|c} \text{energy density} & \text{energy flow} \\ \hline \text{momentum density} & \text{momentum flow} \end{array}\right) = \rho_0\, V^\alpha\, V^\beta$$

In the local rest frame:

$$T^{\alpha\beta} = \left(\begin{array}{c|ccc} \rho_0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Maxwell:

$$\begin{cases} \nabla \cdot \boldsymbol{E} = \rho \\ \nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = 0 \\ \nabla \times \boldsymbol{B} - \frac{\partial \boldsymbol{E}}{\partial t} = \boldsymbol{j} \\ \nabla \cdot \boldsymbol{B} = 0 \end{cases}$$

We can put $\boldsymbol{E}$ and $\boldsymbol{B}$ together:

$$F^{\alpha\beta} = \left(\begin{array}{c|ccc} 0 & E_1 & E_2 & E_3 \\ \hline -E_1 & 0 & B^3 & -B^2 \\ -E_2 & -B^3 & 0 & B^1 \\ -E_3 & B^2 & -B^1 & 0 \end{array}\right)$$

[Actually, while reading through my notes, I found that I had called the above $F_{\alpha\beta}$. I am unsure of whether Martin wrote $F_{\alpha\beta}$ or $F^{\alpha\beta}$ — I could have copied it down wrong — but to be consistent with Weinberg's book, this has to be $F^{\alpha\beta}$.]

$$\partial_\beta F^{\alpha\beta} = J^\alpha \quad \text{gives } \nabla \cdot \boldsymbol{E} = \rho \text{ and } \nabla \times \boldsymbol{B} - \frac{\partial \boldsymbol{E}}{\partial t} = \boldsymbol{j}.$$

$$\varepsilon^{\alpha\beta\gamma\delta} \cdot \partial_\beta F_{\gamma\delta} = 0 \text{ gives the other ones.} \quad (\varepsilon^{0123} = +1)$$

$$A_\alpha = (\phi, A_i); \quad F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \Rightarrow \varepsilon^{\alpha\beta\gamma\delta} \cdot \partial_\beta F_{\gamma\delta} = 0$$

$$(\nabla \times \boldsymbol{v})^i = \varepsilon^{ijk} \partial_j v_k$$