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Course home page: ⟨`http://fy.chalmers.se/~tfemc/Gravitation`⟩. There will be home assignments published on the home page. Deadlines (roughly) fourth and seventh week for the home assignments, some time in January for the home exam.

Newton: description ("I'm not calling it a theory") of static gravity.

$$F = \frac{m_1 m_2 G}{r^2}$$

Newton realized that this was not enough. If one body is moved, when does the other know? Instantly? After some time? Newton did not solve this problem, but he realized that it was there. That is one of the questions that provides us with our main motivation (apart from the other faults of the Newtonian description).

— What if bodies move?

Does the gravitational influence travel with the speed of light? [Yes.] Note the similarity between $F = G\ m_1 m_2 / r^2$ and the electrostatic force of the Coulomb law. Electrodynamics cannot be derived from the Coulomb law, we need to take the step to Maxwell's equations. We want to do something similar for gravity.

• <u>We need a field theory for gravity!</u>

Special relativity

Galileo:

$$\begin{cases} x' = x - v\,t \\ y' = y \\ z' = z \\ t' = t \end{cases}$$

This is not the way nature works. (It is, approximately, when velocities are low.)

Einstein: Lorentz transformations

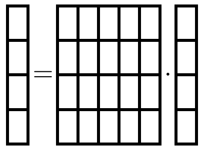
$$\begin{cases} x' = \gamma(v)\,(x - v\,t) \\ y' = y \\ z' = z \\ t' = \gamma(v)\left(t - \frac{v}{c^2}\,x\right) \end{cases}, \quad \gamma(v) = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

This is just a special case, when one system moves along the x axis, but it contains all we need to know. $\gamma(v)$ goes from 0 at $v = 0$ to $+\infty$ as $v \to c$.

We will use a much more compact notation. The first thing we want to do, and then keep it that way throughout the course, is to choose our units in the natural way: $c = 1$. If a year is my unit for time, a light-year is my unit for length. The symmetry between x and t is also clearer now: $x' = \gamma\,(x - v\,t)$; $t' = \gamma\,(t - v\,x)$. This is a classical way of writing it; you have to choose a coordinate system, with axes and so on. We want to use vectors, with four components, since we think of space-time as a four-dimensional space, where time is one of the components. Saving the μ, ν, ... for other things we'll get to later, we will use the beginning of the Greek alphabet, α, β, γ... as indices: x^α, $\alpha = 0, 1, 2, 3$. (For the space part: $\boldsymbol{r} \to x^i$, $i = 1, 2, 3$.) $\alpha = 0$ would normally be the time component, and we normally work in four-dimensional space-time.

The Lorentz transformation is a linear transformation of the coordinates. v is not a function of anything in this context. It is to be regarded as a parameter that relates the two inertial frames. It is just a constant.

$$x'^{\alpha} = \Lambda^{\alpha}{}_{\beta}\, x^{\beta} = \sum_{\beta=0}^{3} \Lambda^{\alpha}{}_{\beta}\, x^{\beta}; \quad \text{in matrix form: } x' = \Lambda\, x;$$

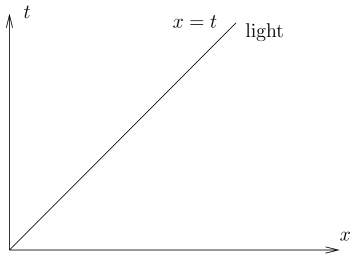


I remind you of a convention that we will use all the time: when one index (here β) is repeated, once downstairs, once upstairs, summation is implied. If an index appears twice upstairs, we have done something wrong.

It is a linear transformation, but it is not *any* linear transformation. If this were a general transformation, we would have 16 constants, but we only have one: v.

There are several ways to do this, perhaps the easiest is to... take a break.

There are two postulates involved: Physics is independent of the coordinate system you use to describe it, and light travels with a constant velocity (in our units, $c = 1$). Light moves one light-second per second, independent of the observer.



More general: $\boldsymbol{r} = \boldsymbol{r}_0 + (t - t_0)\, \boldsymbol{e}$, $\Delta\boldsymbol{r} = \boldsymbol{r} - \boldsymbol{r}_0$, $\Delta t = t - t_0$. Light ray:

$$(\Delta\boldsymbol{r})^2 - (\Delta t)^2 = 0$$

$$\Delta s^2 \equiv (\Delta\boldsymbol{r})^2 - (\Delta t)^2$$

Δs^2 is not really a square, it is a symbol that denotes $(\Delta\boldsymbol{r})^2 - (\Delta t)^2$.

General relativity is about curved spaces, so finite distances are not trivial. We will use infinitesimal distances — if we want something finite we can always integrate it.

Often (In this course, often will mean always): $\mathrm{d}s^2 = |\mathrm{d}\boldsymbol{r}|^2 - \mathrm{d}t^2$.

Light rays: $\mathrm{d}s^2 = 0$. (There is no analogy in Euclidean geometry.) We turn light to geometry. We restrict ourselves to transformations that give $\mathrm{d}s'^2 = \mathrm{d}s^2 = 0$.

(Just as a parenthesis: ["I couldn't spell Einstein, so perhaps I can spell Euclid"] Euclidean geometry: $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = \mathrm{d}\boldsymbol{r}^T\, \mathrm{d}\boldsymbol{r}$. OK to make orthogonal transformations, $\boldsymbol{r}' = P\boldsymbol{r}$, $P^T P = 1$. $\mathrm{d}s'^2 = (P\, \mathrm{d}\boldsymbol{r})^T (P\, \mathrm{d}\boldsymbol{r}) = \mathrm{d}\boldsymbol{r}^T P^T P\, \mathrm{d}\boldsymbol{r} = \mathrm{d}s^2$).

Einstein:

$$\mathrm{d}x' = \Lambda\, \mathrm{d}x$$

$$\mathrm{d}s^2 = |\mathrm{d}\boldsymbol{r}|^2 - \mathrm{d}t^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 - \mathrm{d}t^2$$

$$\mathrm{d}s^2 = \mathrm{d}x^T \eta\, \mathrm{d}x = \eta_{\alpha\beta}\, \mathrm{d}x^{\alpha}\, \mathrm{d}x^{\beta}, \quad \eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

η is the metric of Minkowski space-time (flat space-time).

$$\mathrm{d}s'^2 = (\Lambda\,\mathrm{d}x)^T \eta\,\Lambda\,\mathrm{d}x = \mathrm{d}x^T\,\Lambda^T\,\eta\,\Lambda\,\mathrm{d}x$$

In order to get $\mathrm{d}s'^2 = \mathrm{d}s^2$ in the general case, we need $\eta = \Lambda^T\,\eta\,\Lambda$.

$$\eta_{\gamma\delta}\,\Lambda^\gamma{}_\alpha\,\Lambda^\delta{}_\beta = \eta_{\alpha\beta}$$

(Derive this, as a first homework, and check that the Lorentz transformations given above fulfil this.)

4-vector v^α is a collection of numbers that behaves in a certain way when you do a Lorentz transformation. A 4-vector v^α transforms as $\mathrm{d}x^\alpha$:

$$v'^\alpha = \Lambda^\alpha_\beta\, v^\beta, \quad \Lambda^\alpha_\beta = \frac{\partial x'^\alpha}{\partial x^\beta}, \quad \mathrm{d}x'^\alpha = \Lambda^\alpha_\beta\,\mathrm{d}x^\beta$$

Tensors $t'^{\alpha_1,\ldots,\alpha_n} = \Lambda^{\alpha_1}{}_{\beta_1} \ldots L^{\alpha_n}{}_{\beta_n}\, t^{\beta_1,\ldots,\beta_n}$.

Lowering indices: $v_\alpha \equiv \eta_{\alpha\beta}\, v^\beta$ ($\Rightarrow v^2 \equiv \eta_{\alpha\beta}\, v^\alpha\, v^\beta = v_\alpha\, v^\alpha$).

Raising is done with the inverse of η: $v^\alpha = \left(\eta^{-1}\right)^{\alpha\beta} v_\beta = \eta^{\alpha\beta}\, v_\beta$.