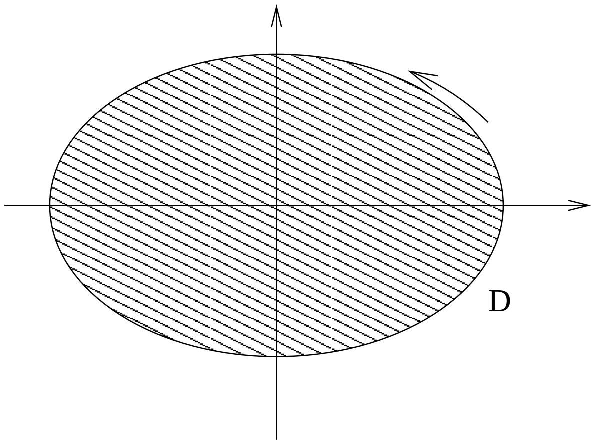
Övning 2006–02–16

EXEMPEL PÅ GREEN: Bestäm arean av ellipsen

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \leqslant 1$$



Lösning:

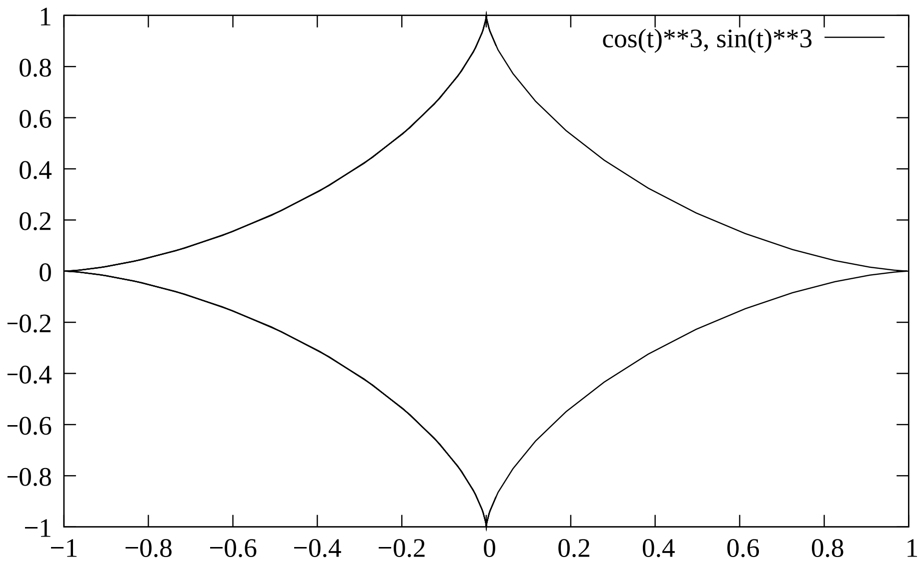
$$m(D) = \iint_D \mathrm{d}x\,\mathrm{d}y = \frac{1}{2}\int_{\partial D} -y\,\mathrm{d}x + x\,\mathrm{d}y =$$

$$= \left[\text{polära koordinater:}\, \partial D\colon \begin{cases} x = a\cos\varphi, & \mathrm{d}x = -a\sin\varphi\,\mathrm{d}\varphi \\ y = b\sin\varphi, & \mathrm{d}y = b\cos\varphi\,\mathrm{d}\varphi \end{cases}, 0 \xrightarrow{\varphi} 2\pi\right] =$$

$$= \frac{1}{2}\int_0^{2\pi} (a\,b\sin^2\varphi + a\,b\cos^2\varphi)\mathrm{d}\varphi = \frac{1}{2}a\,b\int_0^{2\pi} \mathrm{d}\varphi = a\,b\,\pi$$

Asteroid: $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$

$$\begin{cases} x = \cos^3\varphi \\ y = \sin^3\varphi \end{cases}, 0 \xrightarrow{\varphi} 2\pi$$



$$\iint_D \mathrm{d}x\,\mathrm{d}y = \frac{1}{2}\int_{\partial D} -y\,\mathrm{d}x + x\,\mathrm{d}y = \frac{1}{2}\int_0^{2\pi} \left(\sin^3\varphi \cdot 3\cos^2\varphi \cdot \sin\varphi + \cos^3\varphi \cdot 3\sin^2\varphi \cdot \cos\varphi\right)\mathrm{d}\varphi =$$

$$= \frac{3}{2}\int_0^{2\pi} \sin^2\varphi\cos^2\varphi\,\mathrm{d}\varphi = \frac{3}{8}\int_0^{2\pi} \frac{1-\cos 4\varphi}{2}\,\mathrm{d}\varphi = \frac{3}{8}\cdot\frac{1}{2}\cdot 2\pi = \frac{3\pi}{8}$$

$$\left[\sin^2\varphi\cos^2\varphi = \frac{(\sin 2\varphi)^2}{4}\right]$$

Kapitel 8, övning 15.

<fig37>

a) $Y =$ cylinderns mantelyta:

$$\begin{cases} x = 2\cos\varphi \\ y = 2\sin\varphi \\ z = t \end{cases}, (\varphi, t) \in D\colon \begin{cases} 0 \leqslant \varphi \leqslant 2\pi \\ 0 \leqslant t \leqslant 3 \end{cases}$$

$$m(Y) = 4\pi \cdot 3 = 12\pi$$

Med parameteriseringen:

$$m(Y) = \iint_D |\boldsymbol{r}'_\varphi \times \boldsymbol{r}'_t| \mathrm{d}\varphi \mathrm{d}t$$

$$\boldsymbol{r}'_\varphi \times \boldsymbol{r}'_t = \begin{vmatrix} \cdot & \cdot & \cdot \\ -2\sin\varphi & 2\cos\varphi & 0 \\ 0 & 0 & 1 \end{vmatrix} = (2\cos\varphi, 2\sin\varphi, 0)$$

i $(1, \sqrt{3}, 2) = \boldsymbol{r}(\frac{\pi}{3}, 2)$ är $\boldsymbol{n} = (1, \sqrt{3}, 0)$.

$$m(Y) = \iint_D |\boldsymbol{r}'_\varphi \times \boldsymbol{r}'_t| \mathrm{d}\varphi \mathrm{d}t = \iint_D 2\mathrm{d}\varphi \mathrm{d}t = 2\int_0^3 \left(\int_0^{2\pi} \mathrm{d}\varphi\right) \mathrm{d}t = 2 \cdot 3 \cdot 2\pi$$

$$Y_T\colon \begin{cases} y = y(x, z) = \sqrt{4 - x^2} \\ x = x \\ z = z \end{cases} : (y, z) \in D_T\colon \begin{cases} 0 \leqslant z \leqslant 3 \\ 0 \leqslant y \leqslant \sqrt{4 - x^2} \end{cases} : \boldsymbol{r}(x, z) = (x, \sqrt{4 - x^2}, z)$$

$$\boldsymbol{r}'_x \times \boldsymbol{r}'_z = \begin{vmatrix} \cdot & \cdot & \cdot \\ 1 & \frac{-x}{\sqrt{4 - x^2}} & 0 \\ 0 & 0 & 1 \end{vmatrix} = \left(-\frac{x}{\sqrt{4 - x^2}}, -1, 0\right)$$

i $(1, \sqrt{3}, 1) = \boldsymbol{r}(1, 1)$.

$$\boldsymbol{n}_+ = \left(-\frac{1}{\sqrt{3}}, -1, 0\right)$$

Kapitel 10, övning 9

$\boldsymbol{F} = (4xz, -y^2, yz)$

a) <fig38>

$$Y\colon \begin{cases} x = 1 \\ 0 \leqslant y \leqslant 1 \\ 0 \leqslant z \leqslant 1 \end{cases}, \quad F = \iint_Y \boldsymbol{F} \cdot \boldsymbol{N} \, \mathrm{d}S =$$

$$= \iint_Y (4z, -y^2, yz) \cdot (1, 0, 0) \, \mathrm{d}S = \iint_Y 4z \, \mathrm{d}S$$

$$\begin{cases} x = 1 \\ y = y \\ z = z \end{cases} \quad D\colon \begin{cases} 0 \leqslant y \leqslant 1 \\ 0 \leqslant z \leqslant 1 \end{cases}$$

$$= \iint_D 4z \, \mathrm{d}y \, \mathrm{d}z = 4\int_0^1 \int_0^1 z \, \mathrm{d}z \, \mathrm{d}y = 4 \cdot \frac{1}{2} \cdot 1 = 2$$

b) Beräkna flödet av $\boldsymbol{v}$ ut ur kuben

$$K\colon \begin{cases} 0 \leqslant x \leqslant 1 \\ 0 \leqslant y \leqslant 1 \\ 0 \leqslant z \leqslant 1 \end{cases}$$

Svar:

$$F = \iint_Y \boldsymbol{v} \cdot \boldsymbol{n}\, \mathrm{d}S + \iint_{Y_1} \boldsymbol{v} \cdot \boldsymbol{n}_1\, \mathrm{d}S + \iint_{Y_2} \boldsymbol{v} \cdot \boldsymbol{n}_2\, \mathrm{d}S + \iint_{Y_3} \boldsymbol{v} \cdot \boldsymbol{n}_3 \mathrm{d}S +$$

$$+ \iint_{Y_4} \boldsymbol{v} \cdot \boldsymbol{n}_4 \mathrm{d}S + \iint_{Y_5} \boldsymbol{v} \cdot \boldsymbol{n}_5 \mathrm{d}S = (\text{blir för mycket att skriva}) =$$

$$= [\text{Gauss}] = \iiint_K \operatorname{div} \boldsymbol{v}\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z = \iiint_K (4\, z + (-2y) + y) \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z =$$

$$= \int_0^1 \int_0^1 \int_0^1 (4z - y) \mathrm{d}x \mathrm{d}y \mathrm{d}z = \int_0^1 \int_0^1 (4\, z - y) \mathrm{d}z \mathrm{d}y = \int_0^1 (2 - y) \mathrm{d}y = \frac{3}{2}$$

10.21 Beräkna, där $\boldsymbol{n}$ pekar ut ur cylinder med höjd 1. <fig41>

$$\iint_{\partial K} \boldsymbol{v} \cdot \boldsymbol{n} \mathrm{d}S$$

a) $\boldsymbol{v} = (-y, x, 0)$

Flöde genom Y_1, Y_3 är noll, ty $\boldsymbol{v} \| (Y_1, Y_3)$.

Flöde genom Y_2 är också noll, ty $\boldsymbol{v} \|$ mantelytan Y_2.

Eller enklare $F = \int \int \int_K \operatorname{div} \boldsymbol{v}\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z = 0$ då $\operatorname{div} \boldsymbol{v} = 0$.

b) $\boldsymbol{v} = (x, y, 0)$.

Flödet genom Y_1, Y_3 är noll, ty $\boldsymbol{v} \| (Y_1, Y_3)$.

$$F = \iint_{Y_2} \boldsymbol{v} \cdot \boldsymbol{n} \mathrm{d}S = \iint_{Y_2} (x, y, 0) \cdot (x, y, 0) \mathrm{d}S = \iint_{Y_2} (x^2 + y^2) \mathrm{d}S = \iint_{Y_2} \mathrm{d}S = 2\pi$$

Eller med Gauss:

$$F = \iiint_K 2 \mathrm{d}x \mathrm{d}y \mathrm{d}z = 2 \cdot \pi \cdot 1^2 \cdot 1 = 2\pi$$

c) $\boldsymbol{v} = (-y, x, z)$

Gauss:

$$\iiint_K \operatorname{div} \boldsymbol{v}\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z = \iiint_K \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z = m(K) = \pi$$

10.30 <fig42>

$$Y\colon z = x^2, \quad (x, y) \in D, \quad 0 \leqslant y \leqslant x \leqslant 1$$

a) $|\boldsymbol{r}_x' \times \boldsymbol{r}_y'| = \sqrt{1+(z_x')^2+(z_y')^2}$

$$m(Y) = \iint_D \sqrt{1+(z_x')^2+(z_y')^2}\,\mathrm{d}x\,\mathrm{d}z = \iint_D \sqrt{1+4x^2}\,\mathrm{d}x\,\mathrm{d}y =$$

$$= \int_{x=0}^{1} \sqrt{1+4x^2}\left(\int_{y=0}^{x} \mathrm{d}y\right)\mathrm{d}x = \int_0^1 x\sqrt{1+4x^2}\,\mathrm{d}x = \left[\frac{1}{3\cdot 4}(1+4x^2)^{\frac{3}{2}}\right]_0^1 =$$

$$= \frac{1}{12}(5\sqrt{5}-1)$$

b) Flödet av $\boldsymbol{u} = (x, y, z)$ genom Y:

$$Y\colon \begin{cases} x = x \\ y = y \\ z = x^2 \end{cases} \quad (x, y) \in D, \quad \boldsymbol{n} = \boldsymbol{r}_x' \times \boldsymbol{r}_y' = (-z_x', -z_y', 1) = (-2x, 0, 1)$$

$$F = \iint_Y \boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}S = \iint_D (x, y, z)\cdot(-2x, 0, 1)\,\mathrm{d}x\mathrm{d}y = \iint_D \left(-2x^2+x^2\right)\mathrm{d}x\,\mathrm{d}y =$$

$$= \int_0^1 \left(\int_0^x (-x^2)\,\mathrm{d}y\right)\mathrm{d}x = \int_0^1 \left(-x^3\right)\mathrm{d}x = -\frac{1}{4}$$

($\boldsymbol{v}$ strömmar nedåt)