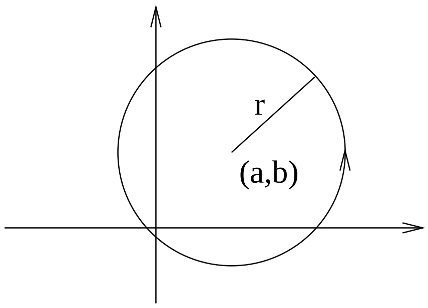
Övning 2006–02–15

9.10 Beräkna

$$\int_\gamma y^2\,\mathrm{d}x + x^2\,\mathrm{d}y$$

där γ är cirkeln $(x-a)^2+(y-b)^2=r^2$ genomlöpt ett varv moturs.



$$D\colon \begin{cases} x-a=\rho\cos\varphi \\ y-b=\rho\sin\varphi \end{cases}, \quad \begin{matrix} 0\leqslant\varphi\leqslant 2\pi \\ 0\leqslant\rho\leqslant r \end{matrix}$$

$$I=\int_\gamma y^2\,\mathrm{d}x+x^2\,\mathrm{d}y=[\text{Green}]=\iint_D (2\,x-2\,y)\,\mathrm{d}x\,\mathrm{d}y=[\text{polärt}]=$$

$$=2\int_0^{2\pi}\int_0^r (a+\rho\cos\varphi-b-\rho\sin\varphi)\,\rho\,\mathrm{d}\rho\mathrm{d}\varphi=$$

$$=2\int_0^r \rho\,[(a-b)\varphi+\rho\sin\varphi+\rho\cos\varphi]_{\varphi=0}^{2\pi}\mathrm{d}\rho=$$

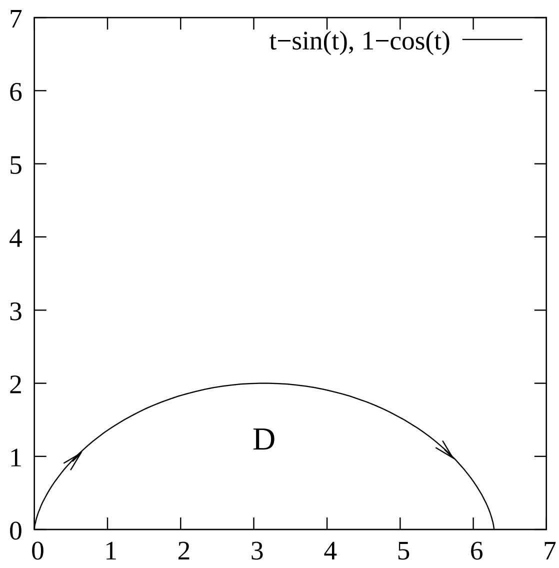
$$=4\,\pi(a-b)\int_0^r \rho\,\mathrm{d}\rho=2\,\pi\,(a-b)\,\rho$$

9.24 Beräkna arean av området mellan x-axeln och cykloidbågen

$$\begin{cases} x=t-\sin t \\ y=1-\cos t \end{cases}, \quad 0\leqslant t\leqslant 2\,\pi$$

$$m(D)=\iint_D \mathrm{d}x\,\mathrm{d}y=\int_{\partial D} Q_x'\mathrm{d}x-P_y'\mathrm{d}y=\int_{\partial D} x\,\mathrm{d}y=\int_{\partial D}-y\,\mathrm{d}x=$$

$$=\frac{1}{2}\int_{\partial D}-y\,\mathrm{d}x+x\,\mathrm{d}y$$



Vänder på genomlöpsriktningen och lägger till den raka sträckan C_1 från $(0,0)$ till $(0,2\pi)$.

$$m(D) = \iint_D \mathrm{d}x\,\mathrm{d}y = \int_{\partial D} -y\,\mathrm{d}x = \int_C -y\,\mathrm{d}x + \int_{C_1} -y\,\mathrm{d}x =$$

$$= \int_C -y\,\mathrm{d}x = -\int_{2\pi}^{0} (1-\cos t)(1-\cos t)\,\mathrm{d}t = \int_0^{2\pi} (1+\cos^2 t - 2\cos t)\,\mathrm{d}t = \frac{3}{2}\,2\,\pi = 3\,\pi$$

$$Y: \boldsymbol{r} = \boldsymbol{r}(u,v), \quad (u,v) \in D$$

$$m(Y) = \iint_Y \mathrm{d}S = \iint_D |\boldsymbol{r}'_u \times \boldsymbol{r}'_v|\,\mathrm{d}u\,\mathrm{d}v$$

8.16 Låt Y vara den sfäriska kalotten <fig8.16>

$$x^2 + y^2 + z^2 = 4, \quad z \geqslant h$$

$$Y: \begin{cases} x = R\sin\theta\cos\varphi & 0 \leqslant \varphi \leqslant 2\pi \\ y = R\sin\theta\sin\varphi & ,\ 0 \leqslant \theta \leqslant \arccos\left(\frac{h}{R}\right) \\ z = R\cos\theta & \end{cases}$$

$$\boldsymbol{r}'_\theta \times \boldsymbol{r}'_\varphi = R\sin\theta \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{Do It!}$$

$$m(Y) = \iint_Y \mathrm{d}S = \int_0^{\arccos\left(\frac{h}{R}\right)} \int_0^{2\pi} |R\sin\theta\,(x,y,z)|\,\mathrm{d}\varphi\,\mathrm{d}\theta =$$

$$= R^2 \int_0^{2\pi} \left(\int_0^{\arccos\left(\frac{h}{R}\right)} \sin\theta\,\mathrm{d}\theta \right) \mathrm{d}\varphi = R^2\,2\,\pi[-\cos\theta]_0^{\arccos\left(\frac{h}{R}\right)} = 2\,\pi R^2\left(1 - \frac{h}{R}\right)$$

Lösning 2:

$$Y: \begin{cases} x = x \\ y = y \\ z = \sqrt{R^2 - x^2 - y^2} \end{cases}, \quad (x,y) \in D = \{(x,y): x^2 + y^2 \leqslant R^2 - h^2\}$$

$$\boldsymbol{r}'_x \times \boldsymbol{r}'_y = (-z'_x, -z'_y, 1) \quad \text{Do It!}$$

$$m(Y) = \iint_Y \mathrm{d}S = \iint_D \sqrt{1 + (z'_x)^2 + (z'_y)^2}\,\mathrm{d}x\,\mathrm{d}y =$$

$$= \iint_D \sqrt{1 + \frac{x^2}{R^2 - x^2 - y^2} + \frac{y^2}{R^2 - x^2 - y^2}}\,\mathrm{d}x\,\mathrm{d}y = \begin{bmatrix} \text{polära} \\ \text{koordinater} \end{bmatrix} =$$

$$= \iint_D \frac{R}{\sqrt{R^2 - x^2 - y^2}}\,\mathrm{d}x\,\mathrm{d}y = \int_{\varphi=0}^{2\pi} \int_{r=0}^{\sqrt{R^2 - h^2}} \frac{R}{\sqrt{R^2 - r^2}}\,r\,\mathrm{d}r\,\mathrm{d}\varphi =$$

$$= 2\,\pi\,R\left[-\sqrt{R^2 - r^2}\right]_0^{\sqrt{R^2 + h^2}} = 2\,\pi\,R(R - h)$$