Övning 2006–02–02

2.42 Visa att ekvationen $x\,y\,z = \arctan(x + y + z)$ lokalt i $(1, -1, 0)$ definierar z som en $\mathcal{C}^1$-funktion av (x, y) och ange en ekvation för tangentplanet till denna yta $z = z(x, y)$ i $(1, -1, 0)$.

Lösning $F(x, y, z) = x\,y\,z - \arctan(x + y + z)$: $(1, -1, 0)$ ligger i nivåytan $F(x, y, z) \equiv 0$.

$$F_z' = x\,y - \frac{1}{1 + (x + y + z)^2}$$

$$F_z'(1, -1, 0) = -1 - 1 = -2 \neq 0$$

Implicita funktionssatsen ger påståendet.

$$F_x' \equiv y\,z - \frac{1}{1 + (x + y + z)^2} = -1$$

$$F_y' \equiv x\,z - \frac{1}{1 + (x + y + z)^2} = -1$$

Tangentplanet:

$$F_x' \cdot (x - 1) + F_y' \cdot (y + 1) + F_z' \cdot (z - 0) = 0$$

$$-(x - 1) - (y + 1) - 2\,z = 0$$

$$x + y - 2z = 0$$

Tillämpningar av integraler

$\rho(x, y, z) =$ densitet (masstäthet). Låt K vara en mätbar kropp i $\mathbb{R}^3$.

1. $M(K) = \int\int\int_K \rho(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ är K:s totala massa.

   $Q(K) = \int\int\int_K q(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ är K:s totala laddning, om q är laddningstäthet.

2. Geometriskt moment av K med avseende på linjen $x = x_0$, hävarm gånger massa:

$$\iiint_K (x - x_0)\,\rho(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

3. Masscentrum (tyngdpunkt): $(x_T,\ y_T,\ z_T)$: geometriskt moment är noll med avseende på $x = x_T$, $y = y_T$ och $z = z_T$.

$$x_T = \frac{1}{M(K)} \iiint_K x\,\rho(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

$$y_T = \frac{1}{M(K)} \iiint_K y\,\rho(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

$$z_T = \frac{1}{M(K)} \iiint_K z\,\rho(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

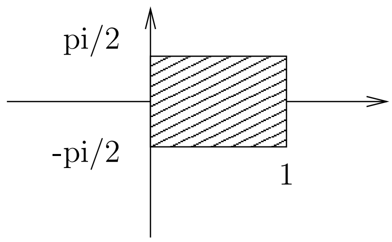
4. Geometriskt tröghetsmoment med avseende på linjen $x = x_0$:

$$\iiint_K (x - x_0)^2 \rho(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

Till exempel det *polära* tröghetsmomentet med avseende på x-axeln:

$$J = \iiint_K (x^2 + y^2)\rho(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

6.4 $\iint_\Delta y \sin(y + x\,y)\,\mathrm{d}x\,\mathrm{d}y$



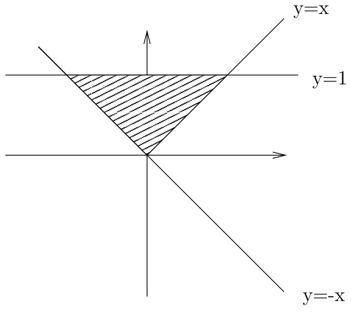
$$\iint_\Delta y \sin(y + x\,y)\,\mathrm{d}x\,\mathrm{d}y = \int_{y=-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{x=0}^{1} y \sin(x + x\,y)\,\mathrm{d}x\,\mathrm{d}y =$$

$$= \int_{y=-\frac{\pi}{2}}^{\frac{\pi}{2}} [-\cos(y + x\,y)]_{x=0}^{1}\,\mathrm{d}y = [\text{jämn}] = 2 \cdot \int_0^{\frac{\pi}{2}} (\cos y - \cos 2\,y)\mathrm{d}y =$$

$$= 2\left[\sin y - \frac{1}{2}\sin 2y\right]_0^{\frac{\pi}{2}} = 2$$

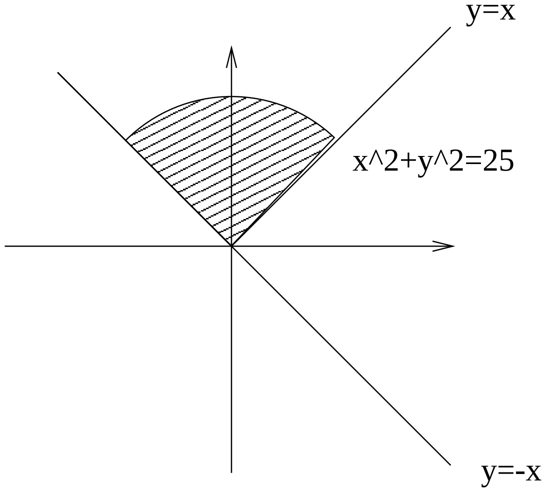
6.14 $D\colon |x| \leqslant y$

$$\iint_D \mathrm{e}^{y^2}\mathrm{d}x\,\mathrm{d}y$$



$$\iint_D \mathrm{e}^{y^2}\mathrm{d}x\,\mathrm{d}y = \int_0^1 \left(\mathrm{e}^{-y^2} \int_{-y}^{y} \mathrm{d}x\right)\mathrm{d}y = \int_0^1 2\,y\,\mathrm{e}^{-y^2}\mathrm{d}y = \left[-\mathrm{e}^{-y^2}\right]_0^1 = 1 - \frac{1}{\mathrm{e}}$$

6.22



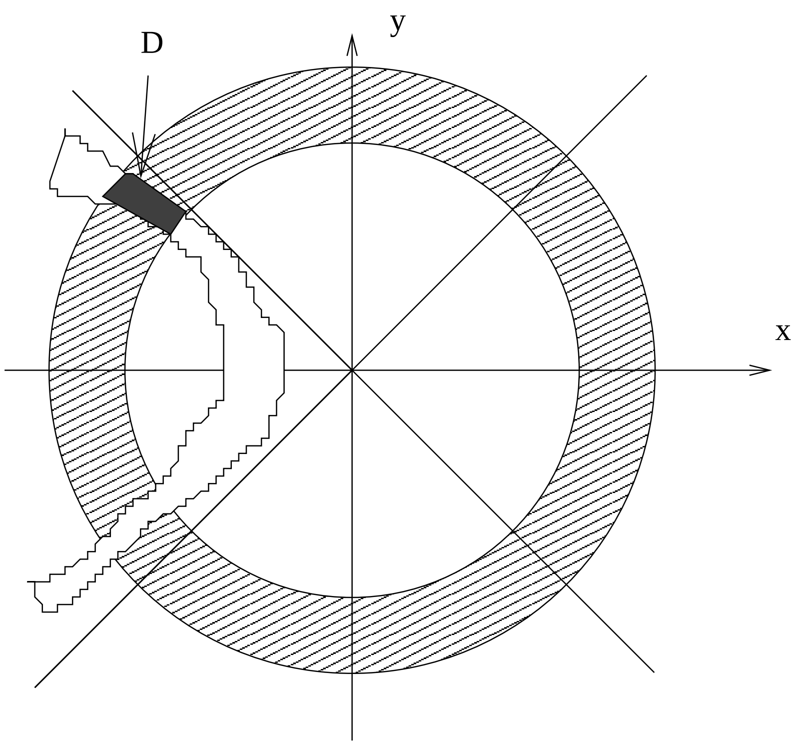
$$\iint_D x^2 \mathrm{e}^{x^2+y^2} = [\text{polära koordinater}] =$$

$$= \int_{\varphi=\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_{r=0}^{5} r^2 \cos^2\varphi \mathrm{e}^{r^2} r \,\mathrm{d}r \,\mathrm{d}\varphi = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{1+\cos 2\varphi}{2} \left(\int_0^5 r^2 \cdot r \cdot \mathrm{e}^{r^2} \mathrm{d}r \right) \mathrm{d}\varphi =$$

$$= \frac{1}{2}\left[\varphi + \frac{1}{2}\sin 2\varphi\right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \cdot \left[\frac{r^2}{2}\mathrm{e}^{r^2} - \frac{1}{2}\mathrm{e}^{r^2}\right]_0^5 = \frac{1}{2}\Big(\frac{\pi}{2} - 1\Big)\left(\frac{24}{2}\mathrm{e}^{25} + \frac{1}{2}\right)$$

$$\left[\int r^2 \cdot r \cdot \mathrm{e}^{r^2} \mathrm{d}r = [\text{partiell integration}] = r^2 \frac{1}{2} \mathrm{e}^{r^2} - \int r e^{r^2} \,\mathrm{d}r\right]$$

6.29



$$\iint_D (x^4 - y^4) \mathrm{d}x \mathrm{d}y$$

Substitution:

$$\begin{cases} u = x^2 - y^2 \\ v = x^2 + y^2 \end{cases} \quad D'\colon \begin{cases} 1 \leqslant u \leqslant 4 \\ \sqrt{17} \leqslant v \leqslant 5 \end{cases}$$

$$(x^4 - y^4) = (x^2 - y^2)(x^2 + y^2)$$

$$\frac{\mathrm{d}(x,y)}{\mathrm{d}(u,v)} = \frac{1}{\frac{\mathrm{d}(u,v)}{\mathrm{d}(x,y)}} = \frac{1}{\begin{vmatrix} 2x & -2y \\ 2x & 2y \end{vmatrix}} = \frac{1}{8\,x\,y} \neq 0 \quad \text{i hela området}$$

$$v^2 - u^2 = 4\,x^2 y^2 \Longrightarrow 8\,x\,y = 4\sqrt{v^2 - u^2}$$

$$\frac{\mathrm{d}(x,y)}{\mathrm{d}(u,v)} = \frac{1}{4\sqrt{v^2 - u^2}}$$

$$\iint_D (x^4 - y^4)\mathrm{d}x\mathrm{d}y = \iint_{D'} u \cdot v \cdot \frac{1}{4\sqrt{v^2 - u^2}}\,\mathrm{d}u\,\mathrm{d}v =$$

$$= \frac{1}{4}\int_1^4 \left(u \int_{\sqrt{17}}^5 \frac{v}{\sqrt{v^2 + u^2}}\mathrm{d}v \right)\mathrm{d}u = \frac{1}{4}\int_1^4 u\left[\sqrt{v^2 - u^2}\right]_{v=\sqrt{17}}^5 \mathrm{d}u =$$

$$= \frac{1}{4}\int_1^4 \left(u\sqrt{25 - u^2} - u\sqrt{17 - u^2} \right)\mathrm{d}u = \frac{1}{4}\left[-\frac{1}{3}(25 - u^2)^{\frac{3}{2}} + \frac{1}{3}(17 - u^2)^{\frac{3}{2}} \right]_1^4 =$$

$$= \frac{1}{12}\left(-27 + 1 - 24\sqrt{24} - 64 \right) = \frac{1}{12}\left(24\sqrt{24} - 90 \right)$$

6.47 $D\colon 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1$

$$\iint_D \frac{1}{\sqrt{|x - y|}}\,\mathrm{d}x\,\mathrm{d}y$$

Generaliserad $(y = x)$. Välj som uttömmande följd: $D_n = A_n \cup B_n$.

$$B_n\colon \begin{cases} 0 \leqslant y \leqslant x - \frac{1}{n} \\ \frac{1}{n} \leqslant x \leqslant 1 \end{cases}$$

$$A_n\colon \begin{cases} 0 \leqslant x \leqslant y - \frac{1}{n} \\ \frac{1}{n} \leqslant y \leqslant 1 \end{cases}$$

$$I_n = \iint_{A_n} \frac{1}{\sqrt{y - x}}\,\mathrm{d}x\,\mathrm{d}y + \iint_{B_n} \frac{1}{\sqrt{x - y}}\,\mathrm{d}x\,\mathrm{d}y =$$

$$= \int_{y=\frac{1}{n}}^1 \int_{x=0}^{y - \frac{1}{n}} \frac{1}{\sqrt{y - x}}\,\mathrm{d}x\,\mathrm{d}y + \int_{x=\frac{1}{n}}^1 \int_{y=0}^{x - \frac{1}{n}} \frac{1}{\sqrt{x - y}}\,\mathrm{d}x\,\mathrm{d}y =$$

$$= 2 \cdot \int_{y=\frac{1}{n}}^1 \int_{x=0}^{y - \frac{1}{n}} \frac{1}{\sqrt{y - x}}\,\mathrm{d}x\,\mathrm{d}y = 2 \cdot \int_{y=\frac{1}{n}}^1 \left[-2\sqrt{y - x} \right]_{x=0}^{y - \frac{1}{n}} \mathrm{d}y =$$

$$= 4\int_{y=\frac{1}{n}}^1 \left(\sqrt{y} - \sqrt{\frac{1}{n}} \right)\mathrm{d}y = 4\left[\frac{2}{3}y^{\frac{3}{2}} - y\sqrt{\frac{1}{n}} \right]_{\frac{1}{n}}^1 =$$

$$= 4\left(\frac{2}{3} - \frac{1}{\sqrt{n}} - \frac{2}{3n\sqrt{n}} + \frac{1}{n} \right)$$

$$I = \lim_{n \to \infty} I_n = \frac{8}{3} \qquad \text{(konvergent)}$$