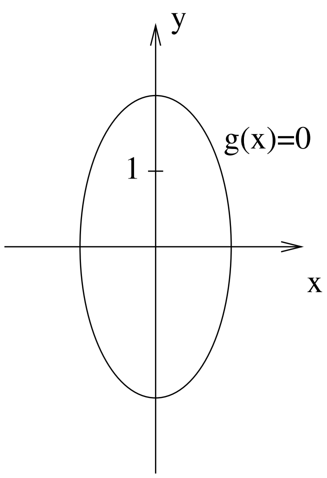
2006–03–02

3 Bestäm max/min av $f(x,y) = (x-2)^2 + y^2 + (x+2)^2 + (y-2)^2$ under bivillkoret

$$C : g(x,y) = x^2 + \frac{y^2}{4} - 1 = 0$$

f är kontinuerlig, ellipsen C är kompakt $\Longrightarrow$ f antar max/min på C.



Lösning (med Lagrange):

$$f(x,y) = 2x^2 + 2y^2 - 4y + 12$$

$$\operatorname{grad} g = \left(2x, \frac{y}{2}\right) \neq (0,0) \text{ på } C, \text{ alltså satisfierar extrempuntkerna:}$$

$$\begin{cases} f'_x = \lambda_0 g'_x \\ f'_y = \lambda_0 g'_y \end{cases} \Longrightarrow \begin{cases} 4x = \lambda_0 2x \\ 4y - 4 = \lambda_0 \frac{y}{2} \end{cases}$$

En möjlighet är $x = 0$, bivillkoret ger punkterna$\pm (0,2)$.

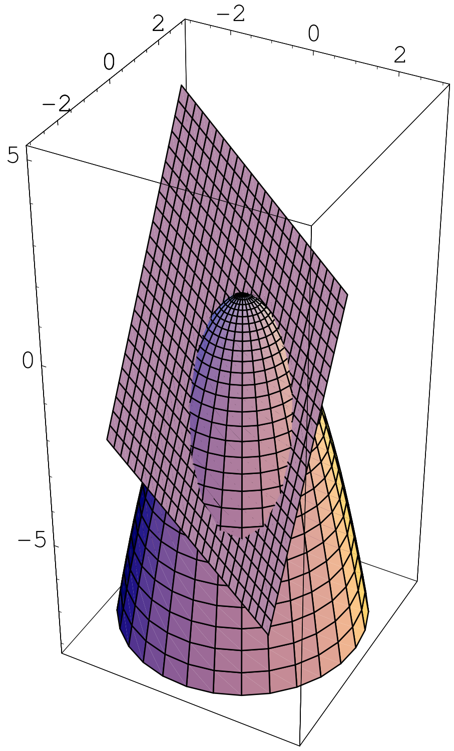
Eller $\lambda_0 = 2 \Longrightarrow y = \frac{4}{3} \Longrightarrow x^2 = 1 - \frac{y^2}{4} = 1 - \frac{16}{9\cdot 4} = \frac{36-16}{36} = \frac{20}{36} = \frac{5}{9}$.

Punkterna blir då $\left(\pm\frac{\sqrt{5}}{3}, \frac{4}{3}\right)$. Max och min finns bland:

$$\begin{cases} f(0,2) = 12 \\ f(0,-2) = 28 \\ f\left(\pm\frac{\sqrt{5}}{3}, \frac{4}{3}\right) = \frac{10}{9} + 2\cdot\frac{16}{9} - \frac{16\cdot 3}{9} + 12 = -\frac{6}{9} + 12 = 11 + \frac{1}{3} \end{cases}$$

Svar: min $\frac{34}{3}$, max 28.

4 Bestäm $\int_\gamma \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r}$. $\gamma = Y_1 \cap Y_2$, $Y_1 : z = 1 - (x^2+y^2)$, $Y_2 : \pi : x - 2y + z = 1$:



γ är snittet mellan ytorna, går moturs i bilden.

Försök med Stokes sats:

$$\int_\gamma \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \iint_Y \operatorname{rot} \boldsymbol{F} \cdot \boldsymbol{n} \mathrm{d}S$$

$$\boldsymbol{F} = \left(\frac{xz}{\sqrt{1+x^2+y^2}} + xz, \frac{yz}{\sqrt{1+x^2+y^2}} - xy, \sqrt{1+x^2+y^2} + xy \right)$$

$$\operatorname{rot} \boldsymbol{F} = (x, x-y, -y)$$

Låt Y vara den del av planet π som begränsas av γ.

$$\iint_Y \operatorname{rot} \boldsymbol{F} \cdot \boldsymbol{n} \mathrm{d}S = \left[\begin{array}{l} x, y \text{ som parametrar} \\ Y : \begin{cases} x = x \\ y = y \\ z = -x + 2y + 1 \end{cases} , (x,y) \in D \end{array} \right] =$$

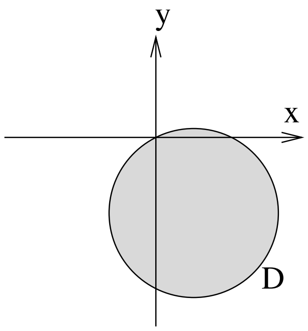
$$= \iint_D (x, x-y, -y) \cdot (1, -2, 1) \mathrm{d}x \mathrm{d}y$$

Skärningen $Y_1 \cap Y_2$:

$$z = 1 - (x^2 + y^2) = 1 - x + 2y$$

$$x^2 + y^2 - x + 2y = 0$$

$$\left(x - \frac{1}{2}\right)^2 + (y+1)^2 = \frac{5}{4}$$



Med polära koordinater:

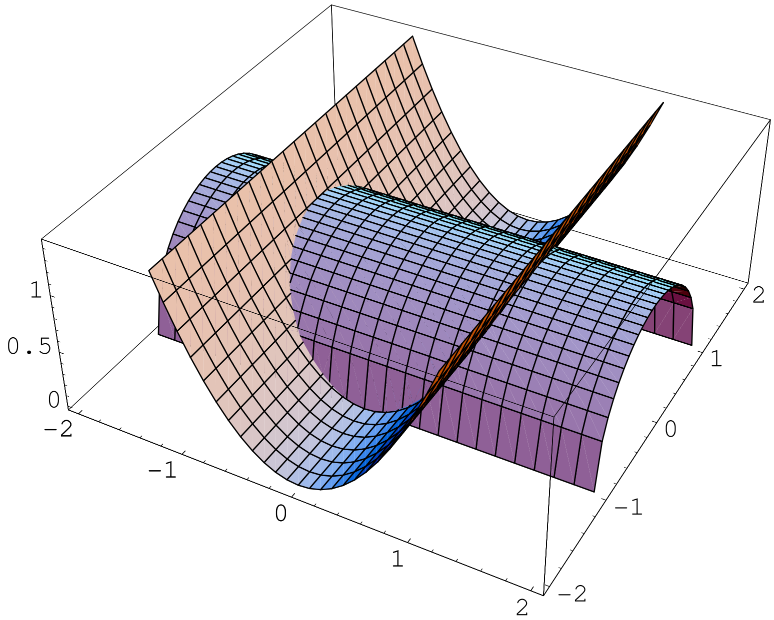
$$\begin{cases} x - \frac{1}{2} = r\cos\varphi \\ y + 1 = r\sin\varphi \end{cases}, \quad D' : \begin{cases} 0 \leqslant r \leqslant \frac{\sqrt{5}}{2} \\ 0 \leqslant \varphi \leqslant 2\pi \end{cases}$$

$$\iint_D (x, x-y, -y) \cdot (1, -2, 1) \mathrm{d}x\mathrm{d}y =$$

$$= \iint_D (y - x)\mathrm{d}x\mathrm{d}y = \int_{\varphi=0}^{2\pi} \int_{r=0}^{\frac{\sqrt{5}}{2}} \left(-1 + r\sin\varphi - \frac{1}{2} - r\cos\varphi\right) r\mathrm{d}r\mathrm{d}\varphi =$$

$$= 2\pi \int_0^{\frac{\sqrt{5}}{2}} \left(-\frac{3}{2}\right) r\mathrm{d}r = -\frac{3}{2} \cdot \pi \cdot \frac{5}{4}$$

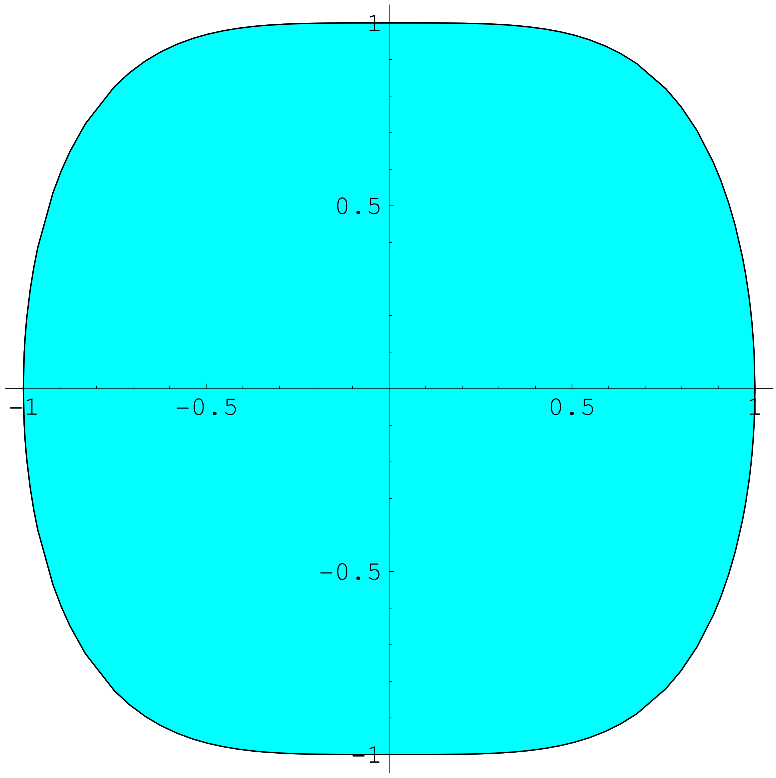
6 K är begränsas av ytorna $z = x^2$ och $z = \sqrt{1-y^2}$:



$$z^2 = x^4 = 1 - y^2$$

$$-\sqrt{1-x^2} \leqslant y \leqslant \sqrt{1+x^4}$$

Projektionen av K på x-y-planet blir mängden D :



Beräkna, med $\boldsymbol{N}$ utåt:

$$\iint_{\partial K} \boldsymbol{F} \cdot \boldsymbol{N} \mathrm{d}S \overset{\text{Försök}}{\underset{\text{Gauss}}{=}} \iiint_K \operatorname{div} \boldsymbol{F} \mathrm{d}x\mathrm{d}y\mathrm{d}z =$$

$$= \left[\boldsymbol{F} = \left(0, yz\sqrt{1-x^4}, \sqrt{1-x^4}\right) \right] =$$

$$= \iiint_K \left(\frac{\partial}{\partial x} 0 + \frac{\partial}{\partial y} \left(yz\sqrt{1-x^4} \right) + \frac{\partial}{\partial z} \left(\sqrt{1-x^4} \right) \right) \mathrm{d}x\mathrm{d}y\mathrm{d}z =$$

$$= \iiint_K z\sqrt{1-x^4} \mathrm{d}x\mathrm{d}y\mathrm{d}z = \int\int_D \left(\sqrt{1-x^4} \int_{x^2}^{\sqrt{1-y^2}} z\mathrm{d}z \right) \mathrm{d}x\mathrm{d}y =$$

$$= \frac{1}{2} \iint_D \sqrt{1-x^4} \cdot \left(1-y^2-x^4\right) \mathrm{d}x\mathrm{d}y = \left[\begin{array}{l} \text{jämn funktion med} \\ \text{avseende på } y \end{array} \right] =$$

$$= \frac{1}{2} \cdot 2 \int_{-1}^{1} \left(\int_0^{\sqrt{1-x^4}} \left((1-x^4)\sqrt{1-x^4} - \sqrt{1-x^4} \cdot y^2 \right) \mathrm{d}y \right) \mathrm{d}x =$$

$$= \int_{-1}^{1} \left((1-x^4)\sqrt{1-x^4}\sqrt{1-x^4} - \frac{1}{3}\sqrt{1-x^4} \left(1-x^4\right) \sqrt{1-x^4} \right) \mathrm{d}x =$$

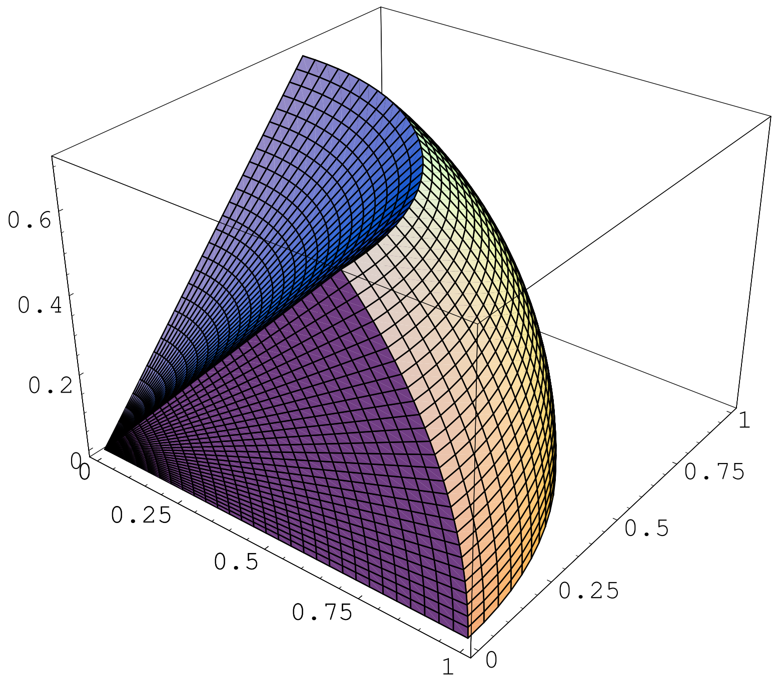
$$= 2\int_0^1 \frac{2}{3}\left(1-x^4\right)^2 \mathrm{d}x = \frac{4}{3}\int_0^1 \left(1+x^8-2x^4\right) \mathrm{d}x = \frac{4}{3}\left(1+\frac{1}{9}-\frac{2}{5}\right)$$

Tenta från 2005–01–14

3 Beräkna

$$\iiint_K \left(xz^2+y^2\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$K : \begin{cases} x \geqslant 0 \\ y \geqslant 0 \\ z \geqslant 0 \end{cases} \quad \text{och} \quad \begin{cases} x^2+y^2+z^2 \leqslant 1 \\ z \leqslant \sqrt{x^2+y^2} \end{cases}$$



Man skulle kunna uttrycka K i rymdpolära koordinater:

$$\begin{cases} x = r\sin\theta\cos\varphi, & 0 \leqslant r \leqslant 1 \\ y = r\sin\theta\sin\varphi, & 0 \leqslant \varphi = \frac{\pi}{2} \\ z = r\cos\theta, & \frac{\pi}{4} \leqslant \theta \leqslant \frac{\pi}{2} \end{cases}$$

$$\iiint_K \left(xz^2+y^2\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z =$$

$$\int_0^{\frac{\pi}{2}} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^1 (r^3\sin\theta\cos\varphi\cdot\cos^2\theta + r^2\sin^2\theta\sin^2\varphi) r^2\sin\theta \mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi =$$

$$= \frac{1}{6}\int_0^{\frac{\pi}{2}}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(\sin^2\theta\cos^2\theta\cdot\cos\varphi\right)\mathrm{d}\theta\mathrm{d}\varphi + \frac{1}{5}\int_0^{\frac{\pi}{2}}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(1-\cos^2\theta\right)\sin\theta\sin^2\varphi\mathrm{d}\theta\mathrm{d}\varphi$$

$$\sin^2\theta\cos^2\theta = \frac{\sin^2 2\theta}{4} = \frac{1-\cos 4\theta}{8}$$

4 Sök max/min av

$$f(y,z) = 10 - 2y^2 - 2z + y^2 + z^2 = (z-1)^2 - y^2 + 9$$

på $D : y^2 + z \leqslant 5$.

6 div $= 0$

$$\iint_Y \boldsymbol{F}\cdot\boldsymbol{n}\mathrm{d}S = \iiint_K = 0 \text{ **Fel!** inte } \mathcal{C}^1$$

$$\iint_Y \boldsymbol{F}\cdot\boldsymbol{n}\mathrm{d}S = \int\int_{Y_\varepsilon = \partial K_\varepsilon}$$

K_ε är ett litet klot. På ellips $\backslash K_\varepsilon$ gäller Gauss.

5

$$z = x^2 + y^2$$

$$\left(\frac{x}{2}\right)^2 + y^2 = 1$$

$$\int_\gamma \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r} \overset{\text{Stokes}}{=} \iint \operatorname{rot}\boldsymbol{F}\cdot\boldsymbol{n}\mathrm{d}S$$

$$Y : \begin{cases} x = x \\ y = y \\ z = x^2y^2 \end{cases}$$

$$\boldsymbol{n} = (-2xy^2, -2x^2y, 1)$$

$$\int_\gamma \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r} = \iint_D (1,1,1)\cdot(-2xy^2, -2x^2y, 1)\mathrm{d}x\mathrm{d}y = -m(D) = -2\pi$$