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ϕ vector (n components). $\delta\phi = \mathrm{i}\,\alpha^a T^a \phi$ where the T^a are $n \times n$ matrices, and α^a is a real parameter. We always assume that the T's form a Lie algebra:

$$\left[T^a, T^b\right] = \mathrm{i}\, f^{abc} T^c$$

What can go wrong if this fails? Suppose we want to exponentiate this (going from infinitesimal to finite).

$$g(\alpha) = \mathrm{e}^{\mathrm{i}\alpha^a T^a}, \quad \phi \to g(\alpha)\phi$$

$$g(\alpha) g(\beta) = g(\gamma(\alpha, \beta))$$

If all the T^a commute

$$\mathrm{e}^{\mathrm{i}\alpha^a T^a} \cdot \mathrm{e}^{\mathrm{i}\beta^a T^a} = \mathrm{e}^{\mathrm{i}(\alpha^a + \beta^a) T^a} \equiv \mathrm{e}^{\mathrm{i}\gamma^a T^a}.$$

If T^a do not commute, $\gamma(\alpha, \beta)$ gets a correction.

$$\gamma^a = \alpha^a + \beta^a + \ldots f^{abc} \alpha^a \beta^c + \cdots$$

$$\delta\phi = \varepsilon\psi$$

$$\delta\psi_\alpha = \mathrm{i}\,\varepsilon^{\dot{\alpha}\dagger} \sigma^\mu_{\alpha\dot{\alpha}} \partial_\mu \phi \equiv \mathrm{i}\left(\sigma^\mu \varepsilon^\dagger\right)_\alpha \partial_\mu \phi$$

$$\partial \mathcal{L}_{\text{free}} = \partial_\mu K^\mu$$

$$[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}] = \text{translation} + \text{equations of motion}$$

Add an auxiliary field F, which is a complex scalar field, to both the Lagrangian and the transformation properties.

$$\delta F = \mathrm{i}\,\varepsilon^\dagger \bar{\sigma}^\mu \partial_\mu \psi$$

$$\delta\psi_\alpha = \mathrm{i}\,\varepsilon^{\dot{\alpha}\dagger} \sigma^\mu_{\alpha\dot{\alpha}} \partial_\mu \phi \equiv \mathrm{i}\left(\sigma^\mu \varepsilon^\dagger\right)_\alpha \partial_\mu \phi + \varepsilon_\alpha F$$

Off-shell Lagrangian.

$$\mathcal{L}_{\text{free}} = -(\partial_\mu \phi^*) \partial^\mu \phi - \mathrm{i}\,\psi^\dagger \bar{\sigma}^\mu \partial_\mu \psi + F^\dagger F$$

$$\frac{\partial \mathcal{L}}{\partial F} = F^\dagger = 0$$

$$[\phi] = 1, \quad [\psi] = \frac{3}{2}, \quad [\varepsilon] = -\frac{1}{2}, \quad [F] = 2$$

$$\delta\mathcal{L} = -(\partial_\mu \phi^*) \varepsilon\, \partial_\mu \psi - \varepsilon \sigma^\mu \bar{\sigma}^\nu \partial_\nu \psi \partial_\mu \phi^\dagger - \mathrm{i} \psi^\dagger \bar{\sigma}^\mu \varepsilon \partial_\mu F - \mathrm{i}\,\varepsilon\, \sigma^\mu \partial_\mu \psi^\dagger \cdot F + \varepsilon^\dagger \ldots$$

$$[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}] \begin{pmatrix} \phi \\ \psi \\ F \end{pmatrix} = 0, \quad \text{off-shell}$$

$$\delta_{\varepsilon_1} \delta_{\varepsilon_2} \phi = \varepsilon_2^\alpha \delta_{\varepsilon_1} \psi_\alpha = \text{before} + \varepsilon_2 \varepsilon_1 F$$

$$[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}] \phi = [\varepsilon_2^\alpha, \varepsilon_{1\alpha}] F = 0$$

$$\varepsilon_1 \varepsilon_2 = +\varepsilon_2 \varepsilon_1 \text{ because of contractions}$$

$$\delta_{\varepsilon_1} \psi = \mathrm{i}\,\varepsilon_1^{\dot{\alpha}\dagger} \sigma^\mu_{\alpha\dot{\alpha}} \partial_\mu \phi + \varepsilon_{1\alpha} F$$

$$\delta_{\varepsilon_2} \delta_{\varepsilon_1} \psi = \mathrm{i}(\sigma^\mu \varepsilon_1)_\alpha \partial_\mu \varepsilon_2 \psi + \mathrm{i}\,\varepsilon_{1\alpha} \varepsilon_2^\dagger \sigma^\mu \partial_\mu \psi$$

Wess–Zumino

$$\mathcal{L} = \mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}}$$

$$\mathcal{L}_{\text{free}} = -\left(\partial\phi^\dagger\right)(\partial\phi) + \psi^\dagger\bar{\sigma}\partial_\mu\psi + F^\dagger F$$

$$\mathcal{L}_{\text{int}} = -\frac{m}{2}\psi^\alpha\psi_\alpha + \text{complex conjugate} + \frac{y}{2}\phi\psi_\alpha\psi^\alpha + \text{complex conjugate} + c\,\phi^2 F + \phi\phi^* F,\, \phi^2 F^*$$

$$+ b\phi F + \text{complex conjugate} + \phi^* F + a\,F + \gamma\,F^2 + U(\phi, \phi^*)$$

where U is the most general quartic potential.

$$\delta_\varepsilon U = \delta_\varepsilon\phi\,\frac{\partial U}{\partial\phi} + \delta_\varepsilon\phi^*\,\frac{\partial U}{\partial\phi^*} = \varepsilon\psi\,\frac{\partial U}{\partial\phi} + \varepsilon^\dagger\psi^\dagger\,\frac{\partial U}{\partial\phi^*}$$

$$\delta F^2 = 2F\,\mathrm{i}\,\varepsilon^\dagger\bar{\sigma}^{\,\mu}\partial_\mu\psi$$

$$\mathcal{L}_{\text{int}} = -\frac{m}{2}\psi^\alpha\psi_\alpha + \text{complex conjugate} + \frac{y}{2}\phi\psi_\alpha\psi^\alpha + \text{complex conjugate} + c\,\phi^2 F + \phi\not{\phi}^* F,\, \phi^2 \not{F}^*$$

$$+ b\phi F + \text{complex conjugate} + \phi^* \not{F}\; + a\,F + \gamma\,\not{F}^2 + U(\not{\phi}, \phi^*)$$

$$\delta_\varepsilon\Big(-\frac{m}{2}\psi\psi + b\phi F\Big) = -\frac{m}{2}\,\mathrm{i}\left(\sigma^\mu\varepsilon^\dagger\right)_\alpha\partial_\mu\phi\,\psi^\alpha - m\,\varepsilon\psi F + b\varepsilon\psi F + b\phi\mathrm{i}\sigma^\mu\varepsilon\partial_\mu\psi = 0$$

$$\varepsilon\psi\psi\psi = 0$$

$$\mathcal{L} = -\partial_\mu\phi^*\partial^\mu\phi - \mathrm{i}\psi^\dagger\bar{\sigma}^{\,\mu}\partial_\mu\psi + F^*F +$$

$$\Big(-\frac{m}{2}\psi\psi - \frac{m}{2}\phi F - \frac{y}{2}\phi\psi\psi - \frac{y}{2}\phi^2 F\Big) + \text{complex conjugate}$$

F is still auxiliary. Still no $\partial F \rightarrow$ eliminate F by algebra. F only couples to ϕ (not ψ).

$$\mathcal{L}_{\text{Boson}} = -(\partial\phi)^2 + F^*F - \frac{m}{2}\,F\phi - \frac{m^*}{2}\,F^*\phi^* - \frac{y}{2}\phi^2 F - \frac{y^*}{2}\phi^{*2}F^*$$

$$\frac{\partial\mathcal{L}}{\partial F} = 0 = F^* - \frac{m}{2}\phi - \frac{y}{2}\phi^2 \quad\Rightarrow\quad F^* = f(\phi)$$

$$\frac{\partial\mathcal{L}}{\partial F^*} = 0 = F - \frac{m^*}{2}\phi - \frac{y}{2}\,\phi^2 \quad\Rightarrow\quad F = f^*(\phi^*)$$

$$f(\phi) = \frac{m}{2}\phi + \frac{y}{2}\phi^2$$

This gives

$$\mathcal{L} = -(\partial\phi)^2 + |f(\phi)|^2 - |f(\phi)|^2$$

$$\mathcal{L} = -(\partial\phi)^2 - \frac{|m|^2}{4}\,\phi^*\phi + \text{“}\,\phi^3\,\text{”} + \frac{|y|^2}{4}\phi^2\phi^{*2}$$

$$\text{Susy} \Rightarrow \text{mass}(\phi) = m(\psi)$$

$$\text{Yukawa} = -\frac{y}{2}\phi\psi\psi$$

$$\text{Quartic bosonic} = -\left(\frac{y}{2}\right)^2\phi^2\phi^{*2}$$

$$\mathcal{L} = -(\partial\phi)^2 - \mathrm{i}\psi^\dagger\bar{\sigma}^{\,\mu}\partial_\mu\psi + F^*F - \frac{m}{2}\psi\psi - \frac{y}{2}\phi\psi\psi - F\Big(a - \frac{m}{2} - \frac{y}{2}\phi^2\Big) + \text{complex conjugate}$$

Introduce a holomorphic function $W(\phi)$ at most cubic (super potential):

$$W(\phi) = \not{\text{const}} + a\phi + \frac{m}{4}\phi^2 + \frac{y}{6}\phi^3$$

$$L = -(\partial\phi)^2 - \mathrm{i}\psi\bar{\sigma}^{\dagger}\partial\psi + F^{\dagger}F - \frac{1}{2}W''\psi^2 - W'F$$

$$W' = a + \frac{m}{2}\phi + \frac{y}{2}\phi^2$$

$$W'' = \frac{m}{2} + y\phi$$

Generalise to N such $\left(\phi^i, \psi^i, F^i\right), i = 1, \ldots, N$. This triplet is a chiral superfield.

$$\mathcal{L} = -\partial_\mu\phi_i^*\partial^\mu\phi^i - \mathrm{i}\,\psi_i^{\dagger}\,\bar{\sigma}^{\,\mu}\partial_\mu\psi^i + F_i^{\dagger}F^i - \frac{1}{2}\frac{\partial^2 W}{\partial\phi^i\partial\phi^j}\psi^i\psi^j - \frac{\partial W}{\partial\phi^i}F^i + \text{complex conjugate}$$

Superpotential $W(\phi^1, \ldots, \phi^N)$ at most cubic and holomorphic (does not depend on $\phi^{\dagger}$). Going on-shell, integrating out F:

$$\mathcal{L} = -(\partial\phi)^2 - \psi^{\dagger}\not{\partial}\psi - \frac{1}{2}\frac{\partial^2 W}{\partial\phi^i\partial\phi^j}\,\psi^i\psi^j - \left|\frac{\partial W}{\partial\phi^i}\right|^2$$