2009–03–24

Lecturer: Gabriele Ferretti

Scattering $A+B\to C_1+\cdots+C_n, n\geqslant 2$. Transition amplitude.

$$\underbrace{\langle C_1,\boldsymbol{p}_{C_1},s_{C_1};...;C_n,\boldsymbol{p}_{C_n},s_{C_n}|}_{t\simeq+\infty}U(+\infty,-\infty)\underbrace{|A,\boldsymbol{p}_A,s_A;B,\boldsymbol{p}_B,s_B\rangle}_{t\simeq-\infty}=$$

$$=\mathrm{i}\,\mathcal{M}(p_A...p_{C_N})(2\pi)^4\delta^{(4)}(p_{\text{in}}-p_{\text{fin}})$$

The physics resides in $\mathcal{M}$, which can be turned into a cross section. There is one general formula.

$$\mathrm{d}\sigma=\frac{1}{4\sqrt{(p_A\cdot p_B)^2-m_A^2\,m_B^2}}\,|\mathcal{M}|^2\,\mathrm{dLips}_n$$

$$\mathrm{dLips}_n=(2\pi)^4\,\delta^{(4)}(p_{\text{in}}-p_{\text{fin}})\cdot\frac{\mathrm{d}^3\boldsymbol{p}_1}{(2\pi)^3\cdot 2\,E_1}\cdots\frac{\mathrm{d}^3\boldsymbol{p}_n}{(2\pi)^3\cdot 2E_n}$$

Lorentz invariant, though it does not look like it at first sight:

$$\left(E=\sqrt{m^2+\boldsymbol{p}^2}\right)\qquad\frac{\mathrm{d}^3\boldsymbol{p}}{2E}=\mathrm{d}^4p\,\delta^{(1)}\big(p^2-m^2\big)\,\theta(p^0)$$

$n=2$, $p_{\text{in}}=(E_{\text{cm}},\boldsymbol{0})$, $|\boldsymbol{p}_1|=\wp$:

$$\int\mathrm{dLips}_2=\int\,(2\pi)^4\,\delta^{(4)}(p_{\text{in}}-p_1-p_2)\,\frac{\mathrm{d}^3\boldsymbol{p}_1}{(2\pi)^3\cdot 2E_1}\cdot\frac{\mathrm{d}^3\boldsymbol{p}_2}{(2\pi)^3\cdot 2\boldsymbol{E}_2}=$$

$$=\frac{1}{(2\pi)^2}\cdot\frac{1}{4}\int\delta^{(1)}\Big(E_{\text{cm}}-\sqrt{m_1+\wp^2}-\sqrt{m_2^2+\wp^2}\Big)\,\frac{\mathrm{d}^2\Omega\,\wp^2\mathrm{d}\wp}{E_1\,E_2}=$$

From now on $\wp$ solves this thing:

$$=\frac{1}{16\pi^2}\cdot\frac{1}{\left|\frac{\mathrm{d}E_1}{\mathrm{d}\wp}+\frac{\mathrm{d}E_2}{\mathrm{d}\wp}\right|}\cdot\frac{\mathrm{d}^2\Omega\,\wp^2}{E_1\,E_2}=\frac{1}{16\pi^2}\cdot\frac{1}{\left|\frac{\wp}{E_1}+\frac{\wp}{E_2}\right|}\cdot\frac{\mathrm{d}^2\Omega\,\wp^2}{E_1\,E_2}=$$

$$\left[\frac{\partial\sqrt{m^2+\wp^2}}{\partial\wp^2}=\frac{2\wp}{2\sqrt{\cdots}}=\frac{\wp}{E}\right]$$

$$=\frac{1}{16\pi^2}\cdot\frac{\wp\,\mathrm{d}^2\Omega}{|E_1+E_2|}$$

$$\mathcal{L}\to\mathcal{M}\to\sigma$$

The first step, $\mathcal{L}\to\mathcal{M}$ is given by Feynman diagrams.

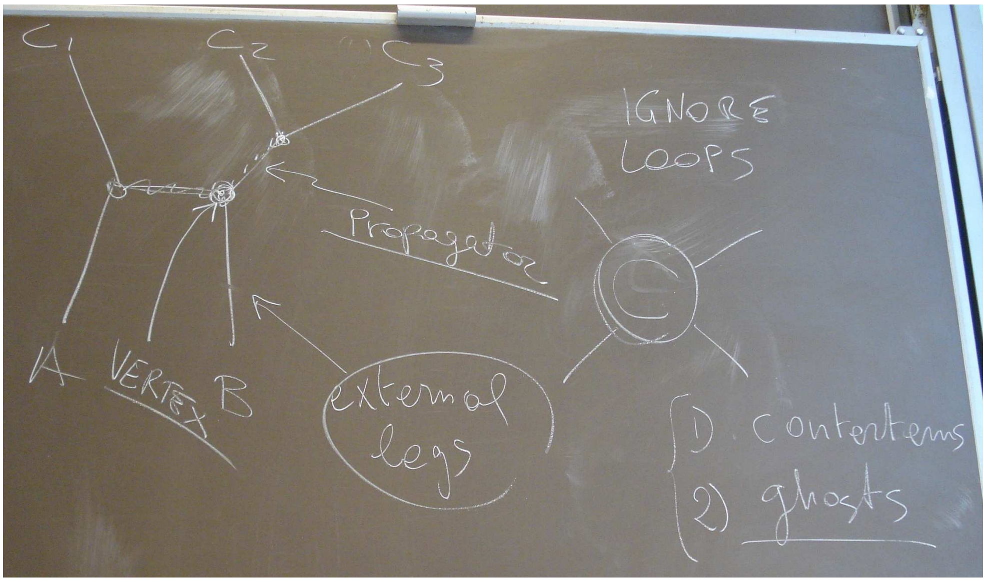

Figure 1.

External legs. The simplest possible case is a spin zero particle. Klein–Gordon:

$$\left(\Box^2 + m^2\right)\phi(x) = 0, \quad \text{where}\, \Box^2 = \partial_\mu\, \partial^\mu = \partial_t^2 - \nabla^2$$

$$\phi(x) = 1 \cdot \mathrm{e}^{\pm \mathrm{i} k x} \Rightarrow k^2 = m^2$$

Spin $\frac{1}{2}$. The Dirac equation:

$$(\mathrm{i}\,\not{\partial} - m)\psi(x) = 0, \quad \text{where}\, \not{\partial} = \gamma^\mu\, \partial_\mu, \quad \{\gamma^\mu, \gamma^\nu\} = 2\,\eta^{\mu\nu}$$

$$\psi(x) = u(p)\,\mathrm{e}^{-\mathrm{i} p x} \quad \text{or}\ \psi(x) = v(p)\,\mathrm{e}^{+\mathrm{i} p x}$$

$$\underbrace{\left(\not{p} - m\right)}_{4\times 4\,\text{matrix}} u = 0$$

$$\left(\not{p} + m\right)\left(\not{p} - m\right) = p^2 - m^2 = 0$$

Spin 1. Maxwell $\partial_\mu F^{\mu\nu} = 0,$ where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$$\Box A_\nu - \partial_\mu \partial_\nu\, A^\mu = 0$$

Plane wave ansatz:

$$A_\mu(x) = \varepsilon_\mu(k)\,\mathrm{e}^{\pm \mathrm{i} k x}$$

$$-\,k^2\,\varepsilon_\nu + k_\nu k \cdot \varepsilon = 0$$

Trivial solution: $\varepsilon_\nu \| k_\nu$, $\varepsilon_\nu \equiv A\, k_\nu$. Gauge equivalent.

Nontrivial solutions require $k^2 = 0$ (the photon is massless) and $k \cdot \varepsilon = 0$ (the photon is transverse). $k^\mu = (k^0, \boldsymbol{k}) = (k, k\,\hat{\boldsymbol{z}})$. Choose a gauge $\varepsilon_0 = 0$. $\varepsilon \cdot k = 0 \Rightarrow \boldsymbol{\varepsilon} \cdot \boldsymbol{k} = 0 \Rightarrow \boldsymbol{\varepsilon} \cdot \hat{\boldsymbol{z}} = 0$. $\varepsilon_{(1)}$, $\varepsilon_{(2)}$.

External legs go into the process and out of the process:

	In	Out
spin 0	1	1
fermions	u^s	$\bar{u}^s$
antifermions	$\bar{v}^s$	v^s
spin 1	$\varepsilon^{r=1,2}$	ε^{r*}

$$\bar{u} = u^\dagger \gamma^0$$

The γ^0 is needed in the Dirac conjugate to make a Lorentz invariant ($\bar{u}u$ is invariant, $u^\dagger u$ is not).

Klein Gordon

$$(\Box + m^2)\phi(x) = J(x) \longleftarrow \text{external source}$$

Green function:

$$(\Box_x + m^2) G(x, y) = -\mathrm{i}\, \delta^{(4)}(x - y)$$

$$\phi(x) = \mathrm{i} \int \mathrm{d}^4 y\, G(x, y)\, J(y)$$

$$(\Box_x + m^2)\phi(x) = (\Box_x + m^2)\, \mathrm{i} \int \mathrm{d}^4 y\, G(x, y)\, J(y) =$$

$$= \mathrm{i} \int \mathrm{d}^4 y\, (\Box_x + m^2) G(x, y)\, J(y) = \mathrm{i} \int \mathrm{d}^4 y \Big[-\mathrm{i}\, \delta^{(4)}(x - y)\Big] J(y) = J(x)$$

$$G(x, y) = G(x - y)$$

$$(\Box + m^2)\, G(x) = -\mathrm{i}\, \delta^{(4)}(x)$$

$$\int \mathrm{d}^4 x\, \mathrm{e}^{\mathrm{i}k\cdot x}\, (\Box + m^2) G(x) = (-k^2 + m^2) \tilde{G}(k)$$

$$\int \mathrm{d}^4 x\, \mathrm{e}^{+\mathrm{i}k\cdot x} (-\mathrm{i}) \delta^{(4)}(x) = -\mathrm{i}$$

$$\tilde{G}(k) = \frac{-\mathrm{i}}{-k^2 + m^2} = \frac{\mathrm{i}}{k^2 - m^2 + \mathrm{i}\varepsilon}$$

The $+\mathrm{i}\varepsilon$ is a prescription to move a pole.

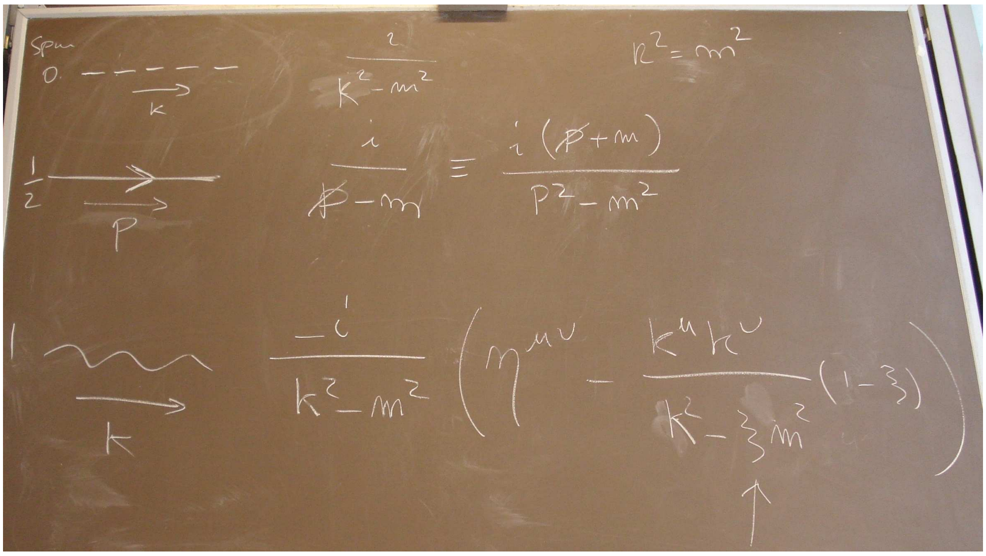

Figure 2. Propagators

$$(\not p-m)(\not p+m)=p^2-m^2$$

$$(\not p-m)(\not p+m)=\not p^2-m^2,$$

$$\not p^2=p_\mu\gamma^\mu p_\nu\gamma^\nu=p_\mu p_\nu\gamma^\mu\gamma^\nu=\frac{1}{2}p_\mu\,p_\nu\gamma^\mu\,\gamma^\nu+\frac{1}{2}\,p_\mu\,p_\nu\,\gamma^\nu\gamma^\mu=p_\mu p_\nu\cdot\frac{1}{2}\{\gamma^\mu,\gamma^\nu\}=p_\mu p_\nu\eta^{\mu\nu}=p^2$$

For the photon propagator, take $m=0,\xi=1$:

$$-\frac{\mathrm{i}\eta^{\mu\nu}}{k^2}$$

The model dependent physics is in the vertices.

Example: Single scalar:

$$\mathcal{L}=\frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi-\frac{m^2}{2}\,\phi^2-\lambda\,\phi^4(x)$$

$$S[\phi]=\int\,\mathrm{d}^4x\,\mathcal{L}(\phi(x),\partial_\mu\phi(x))$$

$\delta S=0\Rightarrow$ Klein–Gordon equation.

$\phi(x)\to\phi(x)+\delta\phi()$, $\delta\phi(x)\to 0$ and $\partial_\mu\delta\phi\to 0$ as $t,\boldsymbol{x}\to\infty$.

$$S[\phi+\delta\phi]=\int\,\mathrm{d}^4x\,\frac{1}{2}(\partial_\mu(\phi+\delta\phi))^2-\frac{m^2}{2}\,(\phi+\delta\phi)^2=S[\phi]+\int\,\mathrm{d}^4x\,\partial_\mu\,\phi\partial^\mu\delta\phi-m^2\phi\delta\phi=$$

$$=\int\,\mathrm{d}^4x\underbrace{\left(-\Box\phi-m^2\phi\right)}_{=0}\delta\phi=0$$

1 scalar ϕ. $h+h\to h+h$.

$$\mathcal{M}=-\,1^4\,\lambda$$

This is a constant amplitude (dimensionless).

$$\mathrm{d}\sigma = \frac{1}{4\sqrt{(p_1 \cdot p_2)^2 - m^4}} \, |\mathcal{M}|^2 \, \mathrm{dLips}_2$$

$$p_1 = \left(\frac{\sqrt{s}}{2}, + \boldsymbol{p} \right), \quad p_2 = \left(\frac{\sqrt{s}}{2}, - \boldsymbol{p} \right)$$

$$\mathrm{d}\sigma = \frac{\lambda^2}{64\pi^2 \, s} \, \mathrm{d}\Omega$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\lambda^2}{64\pi^2 s}$$

QED

$$-\mathrm{i}\, e \bar{\psi} \gamma^\mu \psi A_\mu$$

This term tells us how the electrons interact with the photons.

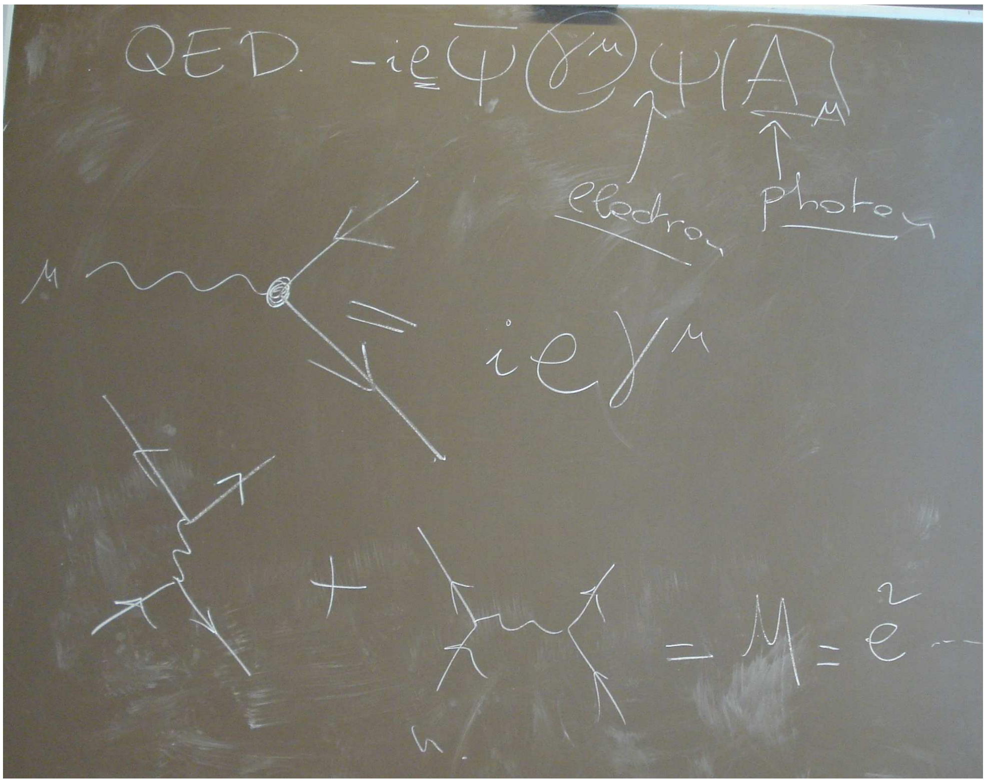

Figure 3.