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$$\rho = \frac{1}{2}\,\dot{\phi}^2 + V(\phi)$$

$$p = \frac{1}{2}\,\dot{\phi}^2 - V(\phi)$$

$$\ddot{\phi} + 3\,H\,\dot{\phi} + V'(\phi) = 0$$

$$H^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\left(\frac{1}{2}\,\dot{\phi}^2 + V(\phi)\right)$$

Initial conditions (at the starting point of inflation, not of the Big Bang itself)

Let's consider a universe of size ~ 1 (in Planck units). What happened before does not matter. Also $\rho \sim 1$. That is,

$$\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\,(\nabla\phi)^2 + V(\phi) \sim 1$$

Typically

$$\frac{1}{2}\,\dot{\phi}^2 \sim \frac{1}{2}(\nabla\phi)^2 \sim V(\phi) = \mathcal{O}(1)$$

If $\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\,(\nabla\phi)^2 \lesssim V(\phi)$ then inflation will begin, and with in $\delta t \sim 1$, both $\frac{1}{2}\,\dot{\phi}^2$ and $\frac{1}{2}(\nabla\phi)^2$ become much smaller than $V(\phi)$.

It thus seems that inflation is quite natural if one can arrange $V(\phi) \sim 1$. Can a universe with $V(\phi) \sim 1$ be created from "nothing"? Need $\Delta E\,\Delta t \lesssim 1$. Total energy in the universe $=$ volume $\times$ $\rho \sim H^{-3}\,\rho \sim V^{-3/2} \times V = V^{-1/2}$. Therefore such a universe can appear quantum mechanically within $\Delta t \gtrsim 1$ if V is not too much smaller than 1.

A specific example: $V(\phi) = \frac{1}{2}\,m^2\,\phi^2$.

The equations become

$$\begin{cases} \ddot{\phi} + 3\,H\,\dot{\phi} = -\,m^2\phi \\ H^2 + \frac{k}{a^2} = \frac{1}{6}\left(\dot{\phi}^2 + m^2\phi^2\right) \end{cases}$$

Here $8\pi G = 1$, $m_{\mathrm{Pl}} = 1$. Assume that initially both H and ϕ are large. This means that initially one has

$$\ddot{\phi} \ll 3\,H\,\dot{\phi}; \quad \dot{\phi}^2 \ll m^2\,\phi^2$$

$$H^2 \gg \frac{k}{a^2}$$

We get

$$H = \frac{\dot{a}}{a} = \frac{m\,\phi}{\sqrt{6}}, \quad \dot{\phi} = -\,m\,\sqrt{\frac{2}{3}}$$

From the first equation we see that if ϕ changes slowly that a grows approximately as e^{Ht} with $H = m\,\phi/\sqrt{6}$. Inflation ends when $\phi < 1$ (m_{Pl}).

In this simple model inflation occurs without complicated assumptions like tunneling from a false vacuum, etc.

How long does inflation last?

Let us consider a potential which is almost constant / flat near $\phi \approx 0$. In this "slow-roll" region we may neglect $\ddot{\phi}$ sot that the equation of motion reduces to $3\,H\,\dot{\phi} = -\,V'(\phi)$. If we also assume that $\frac{1}{2}\,\dot{\phi}^2 \ll V(\phi)$ we obtain

$$H^2 \approx \frac{8\pi G}{3}\,V(\phi)$$

Dividing these equations we find

$$H = -\,8\pi G\,\frac{V(\phi)}{V'(\phi)}\,\dot{\phi}$$

Integrating,

$$N(\phi_1 \to \phi_2) \equiv \ln\left[\frac{a(t_2)}{a(t_1)}\right]$$

$$= \int_{a(t_1)}^{a(t_2)} \mathrm{d}(\ln a) = \int_{t_1}^{t_2} H\,\mathrm{d}t = -\,8\pi G \int_{\phi_1}^{\phi_2} \frac{V(\phi)}{V'(\phi)}\,\mathrm{d}\phi$$

Example: $V(\phi) = V_0\,\phi^k$

$$N(0 > \phi) = -\,\frac{8\pi G}{k}\,\phi^2 = -\,8\,\pi\,G\,\frac{V(\phi)}{V''(\phi)}\,(k-1)$$

One can show that $V''(\phi)/V(\phi) \ll 1$, so $N(0 \to \phi) \gg 1$.

Hybrid inflation

Inflation can also occur in models with more than one scalar field.

Fluctuations

It can be shown that the average amplitude of a fluctuation in ϕ is given by

$$\delta\phi \approx \frac{H}{2\pi}$$

(See eq. E.49.) which implies

$$\delta t = \frac{\delta\phi}{\dot{\phi}} \sim \frac{H}{2\pi\,\dot{\phi}}$$

(delay in the end of inflation).

After inflation $\rho \sim H^2$ where $H \sim t^{-1}$, so a local delay of the end of inflation implies

$$\delta_H \sim \frac{\delta\rho}{\rho} \sim \frac{\delta t}{t} \sim \frac{H^2}{2\pi\dot{\phi}} \sim \frac{m^2\,\phi^2}{2\pi\,6}\,\frac{\sqrt{3}}{\sqrt{2}m} \sim \frac{m\phi^2}{4\sqrt{2}\pi\sqrt{6}}$$

It is known that $\phi_H \sim 15$ and that $\delta_H \sim 10^{-5}$. δ_H measured by COBE satellite, 2006 Nobel price. Fluctuations are very important to test inflation quantitatively.

The cosmic microwave background radiation (CMBR)

As we have discussed before there is an isotropic (to a very high accuracy) microwave background radiation with $T \approx 2.73\,\mathrm{K}$. It was discovered experimentally by Penzias and Wilson.

The background radiation arose as follows: In the early universe electromagnetic radiation (photon) was generated and kept in equilibrium via reactions of the type $e^+ + e^- \to \gamma + \gamma$. In the subsequent expansion the radiation cooled until the threshold for pair production was reached. Before this temperature the radiation was kept in equilibrium through processes such as Compton scattering ($e^- + \gamma \to e^- + \gamma$).

When the temperature fell below the photoionization energy of hydrogen, the reaction $\mathtt{H} + \gamma \rightleftarrows p + e^-$ was no longer in equilibrium and the photons decoupled from matter. This happened at around $T \sim 0.25\,\mathrm{eV}$, $z \sim 1000$. These photons have now been cooled to $2.73\,\mathrm{K}$.

We know from before that the number density of the photons is (using $E = \omega, \hbar = 1$).

$$n(\omega, T) = \frac{1}{\pi^2} \frac{\omega^2 \mathrm{d}\omega}{\mathrm{e}^{\omega/T} - 1}$$

After decoupling, the frequency and temperature have been redshifted via

$$\omega_0 = \frac{\omega}{1+z}, \qquad T_0 = \frac{T}{1+z}$$

Thus

$$n(\omega, T)\,\mathrm{d}\omega = (1+z)^3\, n(\omega_0, T_0)\,\mathrm{d}\omega_0$$

Similarly the energy density is

$$u(\omega, T)\mathrm{d}\omega = \frac{1}{\pi^2} \frac{\omega^3\,\mathrm{d}\omega}{\mathrm{e}^{\omega/T} - 1}$$

This leads to

$$\rho = \int_0^\infty u(\omega, T)\mathrm{d}\omega = \left[x = \frac{\omega}{T}\right] = \frac{T^4}{\pi^2} \int_0^\infty \frac{x^3 \mathrm{d}x}{\mathrm{e}^x - 1} = \frac{\pi^2}{15} T^4$$

This is the familiar Stefan–Boltzmann law.