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The lookback time and the age of the universe

The lookback time is the time difference between an event that occurred at a redshift z and the present time.

First, recall the definition of $H(t)$:

$$H(t) = \frac{\mathrm{d}}{\mathrm{d}t}\ln\left(\frac{a(t)}{a_0}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\ln\left(\frac{1}{1+z}\right) = -\frac{1}{1+z}\frac{\mathrm{d}z}{\mathrm{d}t}$$

Friedmann equation:

$$H^2(t) = \left(\frac{\rho}{\rho_\mathrm{c}} - \frac{k}{H^2 a^2}\right)$$

with $\rho = \rho_m + \rho_\Lambda$ we find (using $\rho_m \propto a^{-3}$)

$$H^2(z) = H_0^2\left(\Omega_m(1+z)^3 + \Omega_k(1+z)^2 + \Omega_\Lambda\right), \text{with}$$

$$\Omega_m = \frac{\rho_m}{\rho_\mathrm{c}}, \quad \Omega_\Lambda = \frac{\rho_\Lambda}{\rho_\mathrm{c}}, \quad \Omega_k = \frac{-k}{a^2 H^2}.$$

Now we have all the pieces that we want. The general discussion is in the book, here we do a specific example.

Example: In a universe with $\Omega_m = 1, \Omega_\Lambda = 0, \Omega_k = 0$, we have

$$t_0 = H_0^{-1}\int_0^\infty \frac{1}{(1+z)^{5/2}}\,\mathrm{d}z = \frac{2}{3}\cdot\frac{1}{H_0}.$$

In a matter-dominated universe, this would be the age of the universe: $t_0 = \frac{2}{3} H_0^{-1}$.

It is important to distinguish between two cases. Note that the age of the universe and the “Hubble time” $1/H_0$ are two distinct concepts.

In general, you get an integral involving all three forms of energy, that you cannot do analytically.

The standard model of particle physics

Quantum field theory is quite complicated and we will need a lot of it, but we won’t go in too deep.

One might think that the atomic and subatomic forces as well as quantum mechanics are not relevant to cosmology, which deals with large scale structure. This is *not* the case. When the universe was younger it was also smaller and hotter and all the forces of nature played an important role. To understand the early universe (and perhaps the Big Bang itself) particle physics is essential. Even today relics from the early universe can be observed (such as the cosmic microwave background radiation).

The particles of the standard model are described as modes of quantised fields. The proper framework is quantum field theory. Quantum field theory is complicated, so we do classical stuff first, to make the quantum stuff easier to understand. Let’s start with the classical analysis using classical field theory.

Forces of nature

Besides gravity and the electromagnetic force, there are two additional known forces: the strong and the weak forces. The strong force is described by a so called "SU(3) gauge theory". The electromagnetic force and the weak interaction forms an "SU(2) × U(1) gauge theory". These forces are mediated by spin one (i.e. vector) particles.

This is what we are aiming for: we are trying to understand what these words mean. We start with the word "gauge":

Gauge theories

In electromagnetism, there is one fundamental field, the gauge field $A_\mu(x)$ with field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, where $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$. We see that $F_{\mu\nu}$ is antisymmetric, which means that it has six independent components. These six components are the familiar electric and magnetic fields, $\boldsymbol{E}$ and $\boldsymbol{B}$, which enter into Maxwell's equations.

$$F_{\mu\nu} \doteq \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B & B \\ \vdots & & 0 & B \\ & & & 0 \end{pmatrix}$$

In this framework, the equations of motion (in vacuum): $\partial^\mu F_{\mu\nu} = 0$. This is Maxwell's equations.

This can be obtained from a Lagrangian [which Niclas spells "lagrangean", which he says is a good way to spell it]:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

with an action $S = \int \mathrm{d}^4x\, \mathcal{L}(x)$. We take the variation $A_\mu \to A_\mu + \delta A_\mu$, and get to linear order

$$S \to S + \int \mathrm{d}^4x\, (\partial_\mu F^{\mu\nu}\, \delta A_\nu).$$

We identify the factor $\partial_\mu F^{\mu\nu}$, which we recognise from the equations of motion $\partial_\mu F^{\mu\nu} = 0$.

To familiarise you with the variational calculus: In Newtonian mechanics we have $L = \frac{m\dot{y}^2}{2} - V(y)$, and $S = \int L\, \mathrm{d}t$. Making a variation in y, $y \to y + \delta y$, we see how S changes:

$$S \to S - \int \mathrm{d}t [m\, \ddot{y}\, \delta y + V'(y)\, \delta y] \quad \Rightarrow \quad m\ddot{y} = -V'(x) = F(x)$$

This is Newton's second law for a conservative force.

I still have not told you what a gauge theory is, but let's do the SU(N) stuff first.

SU(N) is a Lie algebra. You can think of the elements of SU(N), as all hermitean, traceless $N \times N$ matrices, i.e.

$$T \in \mathrm{SU}(N) \quad \Leftrightarrow \quad T^\dagger \equiv \left(T^t\right)^* = T, \quad \mathrm{tr}(T) = 0$$

Dimension of SU(N) $= N^2 - 1$.

Sometimes you see the U(N) Lie algebra. You can think of it as all hermitean $N \times N$ matrices. The dimension of $U(N) = N^2$. U(N) is SU(N) without the traceless condition, like SU(N) + unit matrix.

Some examples:

$\mathrm{U}(1) = \mathbb{R}$

$\mathrm{SU}(2)$: The Pauli matrices $\sigma_x, \sigma_y, \sigma_z$ form a basis.

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

An aside: WMAP is the satellite that measures the cosmic microwave background radiation. See ⟨`http://map.gsfc.nasa.gov/`⟩.

The Lagrangian is easier to guess than the equations of motion. The Lagrangian is a scalar function, and it has to have all the symmetries of the theory. There is not too much to write down. Demand gauge invariance and Lorentz invariance, then you know the Lagrangian.

Generalisation of electromagnetism

Let $\mathbb{A}_\mu = A_\mu^a(x) T_a$ (sum over a), where $T_a \in \mathrm{SU}(N)$, $a = 1, \ldots, N^2 - 1$. We go from ordinary numbers to matrices.

$$\mathbb{F}_{\mu\nu} = \partial_\mu \mathbb{A}_\nu - \partial_\nu \mathbb{A}_\mu + \mathrm{i}\, g[\mathbb{A}_\mu, \mathbb{A}_\nu]$$

g is just a constant.

$$\mathcal{L} = -\frac{1}{4} \mathrm{tr}(\mathbb{F}_{\mu\nu} \cdot \mathbb{F}^{\mu\nu}) \propto -\sum_{a=1}^{N^2-1} F_{\mu\nu}^a F^{a\mu\nu}$$

$$\mathrm{tr}(T^a T^b) \propto \delta^{ab} \quad (\text{verify for } \sigma_{x,y,z})$$

This is $\mathrm{SU}(N)$ Yang-Mills theory. In the 2×2 case $T^1 = \sigma_x, T^2 = \sigma_y, T^3 = \sigma_z$ and

$$\mathbb{A}_\mu = A_\mu^1 \sigma_1 + A_\mu^2 \sigma_y + A_\mu^3 \sigma_z.$$

The standard model Lagrangian contains the terms

$$\mathcal{L}_{\text{gauge}} = \underbrace{-\frac{1}{4} \sum_{a=1}^{8} G_{\mu\nu}^a G^{a\mu\nu}}_{\substack{\mathrm{SU}(3),\, g_3 \\ \text{strong force}}} \underbrace{-\frac{1}{4} \sum_{a=1}^{3} F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu}}_{\substack{\mathrm{SU}(2) \oplus U(1) \\ g_2 \quad g_1 \\ \text{electroweak}}}$$

What is a gauge theory, and what is a gauge transformation? That is what fixes this Lagrangian.

Gauge transformations

The Lagrangian of Maxwell's theory (a U(1) albelian Yang-Mills theory) is invariant under the transformation $A_\mu \to A_\mu + \partial_\mu \alpha(x)$. $(F_{\mu\nu} \to F_{\mu\nu})$

Similarly, the Lagrangian of the (pure) $\mathrm{SU}(N)$ gauge theory is invariant under $\mathbb{A}_\mu \to \mathbb{A}_\mu + \partial_\mu \boldsymbol{\alpha} + \mathrm{i}\, g[\mathbb{A}_\mu, \boldsymbol{\alpha}] \equiv \mathbb{A}_\mu + \mathbb{D}_\mu \boldsymbol{\alpha}$, where $\mathbb{D}_\mu = \mathbb{I}\, \partial_\mu + \mathrm{i}\, g\, [\mathbb{A}_\mu, \cdot\,]$ (covariant derivative).

An important difference between Maxwell's theory and the $\mathrm{SU}(N)$ Yang-Mills theory is that the latter has terms like $\partial \mathbb{A}^3$ and $\mathbb{A}^4$ in the Lagrangian, whereas Maxwell's theory only has the kinetic terms $(\partial A)^2$.

Physically this means that photons (γ) do not interact with each other, whereas the particles which mediate the SU(2) (weak) $W^\pm, Z$, and SU(3) (strong, gluons) *do* interact.

Equation of motion for Yang-Mills $\mathbb{D}^\mu \mathbb{F}_{\mu\nu} = 0$.

Scalar (spin 0) fields

A spin 0 field $\phi(x)$ is described by an $\mathcal{L}$ of the form

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - V(\phi)$$

$V(\phi)$ is a potential. There are some special potentials that are important. A (free) massive particle has $V(\phi) = \frac{1}{2} m^2 \phi^2$ with equations of motion $\partial_\mu \partial^\mu \phi + m^2 \phi = 0$. (This is the Klein-Gordon equation.)

Higher powers in ϕ describe self-interactions.

Note that $\mathcal{L} \propto F_{\mu\nu} F^{\mu\nu}$ describe a massless vector particle (photons have two polarisations). A mass term is forbidden by gauge invariance, since $m^2 A_\mu A^\mu$ is *not* gauge invariant.

A complex scalar field $\phi = \phi_1 + \mathrm{i}\phi_2$ can be coupled to electromagnetism:

$$\mathcal{L} = \frac{1}{2}[(\partial_\mu + \mathrm{i}\, e\, A_\mu)\phi]^* [(\partial_\mu + \mathrm{i}\, e\, A_\mu)\phi] - V(\phi\phi^*) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

This is invariant under $A_\mu \rightarrow A_\mu + \partial_\mu \alpha$, $\phi \rightarrow \mathrm{e}^{-\mathrm{i}e\alpha(x)} \phi$.