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Friedmann equations:

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho$$

$$2\,\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = -\,8\pi G\,p$$

From them follows conservation of energy:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho a^3) = -\,p\,\frac{\mathrm{d}}{\mathrm{d}t}(a^3)$$

- Relativistic matter/energy (radiation) $p = \rho/3$, $\rho \propto a^{-4}$.
- Nonrelativistic matter (dust) $p = 0$, $\rho \propto a^{-3}$.

For small $t, a(t) \sim t^{\alpha}$ is sometimes useful.

- Vacuum energy: $p = -\,\rho$, $\rho = \text{constant}$.

Toy models for FRW spacetime

To get some feeling for the form of the FRW metric one can consider lower-dimensional examples. The two-dimensional sphere S^2 is defined via $x^2 + y^2 + z^2 = a^2$ in $\mathbb{R}^3$. This implies

$$\mathrm{d}z = \frac{-\,x\,\mathrm{d}x - y\,\mathrm{d}y}{z} = \pm\frac{x\,\mathrm{d}x + y\,\mathrm{d}y}{\sqrt{a^2 - x^2 - y^2}}$$

so that

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \frac{(x\,\mathrm{d}x + y\,\mathrm{d}y)^2}{a^2 - x^2 - y^2}.$$

Let $x = a\,r\cos\theta, y = a\,r\sin\theta$ which implies

$$\mathrm{d}s^2 = a^2\left(\frac{\mathrm{d}r^2}{1 - r^2} + r^2\,\mathrm{d}\theta^2\right)$$

Similarly, starting from a hyperboloid embedded in $\mathbb{R}^3$: [I think you are wrong here. I believe in embedding in a Minkowski space, rather than an Euclidean space.]

$$x^2 + y^2 - z^2 = a^2$$

one can show that the metric (on the Lobachevski plane) can be written:

$$\mathrm{d}s^2 = a^2\left(\frac{\mathrm{d}r^2}{1 + r^2} + r^2\,\mathrm{d}\theta^2\right)$$

Note also that the two-dimensional plane $\mathbb{R}^2$ has a metric of the above form,

$$\mathrm{d}s^2 = a^2\left(\frac{\mathrm{d}r^2}{1 - k\,r^2} + r^2\,\mathrm{d}\theta^2\right) \quad \text{with } k = 0.$$

⟨fig⟩

The expansion and Hubble's law

We know from observations that galaxies move away from each other, so we have at the present time $H(t_0) \equiv \dot{a}/a > 0$.

Comment: Since the universe is homogeneous and isotropic the expansion looks the same for all observers. There is a preferred frame (the so-called comoving frame, or cosmic rest frame) in which the microwave background radiation is maximally isotropic.

A two-dimensional toy model

Consider a universe whose spatial part is the surface of a sphere with a radius $a(t)\, r$. A "galaxy" in this universe (with $\theta, r = \text{constant}$) has a velocity

$$v_{\mathrm{AB}} = \frac{\mathrm{d}}{\mathrm{d}t} d_{\mathrm{AB}}(t) = \frac{\mathrm{d}}{\mathrm{d}t} a(t)\, r\, \theta_{\mathrm{AB}}$$

⟨fig⟩ A and B are two points on the sphere.

$$v_{\mathrm{AB}} = \frac{\dot{a}(t)}{a(t)} d_{\mathrm{AB}}(t)$$

Thus $v = H(t)\, d$ with $H(t) = \dot{a}(t)/a(t)$.

Deceleration parameter $q(t) = -\, \ddot{a}/a\, H^2$. In terms of H and q the Friedmann equations can be written

$$1 + \frac{k^2}{a^2 H^2} = \rho/\rho_{\mathrm{c}} \equiv \Omega \quad \left(\rho_{\mathrm{c}} = \frac{3\, H^2}{8\pi G}\right)$$

$$q = \frac{1}{2} (\rho + 3p)/\rho_{\mathrm{c}}$$

Note that $a(t) = a(t_0)\Big(1 + H(t_0)\,(t - t_0) - \frac{1}{2} q(t_0) H(t_0)^2 (t - t_0)^2 + \cdots\Big)$.

The fate of the universe

Since the scale factor $a(t)$ determines whether the universe is expanding or contracting (or just static), it is of interest to investigate what happens as $t \to \infty$.

The continuing redshift of the cosmic microwave background radiation means that radiation can be neglected as $t \to \infty$.. Thus only ρ_m and ρ_Λ play a role. Recall,

$$\dot{a}(t)^2 = \frac{8\pi G \rho_0 a_0^3}{3\, a(t)} - k + \Lambda \frac{a(t)^2}{3}$$

Some examples:

If $\Lambda = 0$ and $k = -\,1, 0$, then the universe will expand forever. If $k = +\,1$ the $\dot{a}(t)$ will be zero when $a(t) = a_{\mathrm{crit}} = 8\pi G\, \rho_0 a_0^3/3$. Since $\ddot{a}(t) = -\, (\dot{a}/a)^2\, a - 1/a^2 \cdot a < 0$, the universe will start to contract after reaching $a = a_{\mathrm{crit}}$. ⟨fig⟩

If $\Lambda \neq 0$ one sees that for small $a(t)$ its effect is negligible, but for large $a(t)$ its effect will dominate. If $\Lambda < 0$, then $\dot{a}$ will again be 0 for some t and since $\ddot{a} < 0$ (now independent of k) the universe will again start to contract (might even oscillate).

If $\Lambda > 0$, and $k = 0$ or $k = -\,1$, then we will have exponential expansion. If $k = +\,1$, the case is more subtle. As an example, there is a static solution. This was the reason why Einstein introduced the cosmological constant, since he wanted the universe to be static.

Horizon distance

The horizon distance is the distance light can travel from the beginning to a particular time t_0. Since for light $\mathrm{d}s=0$ we get for radial motion ($\theta,\varphi=\text{const}$)

$$\mathrm{d}t=\frac{a(t)}{\sqrt{1-k\,r^2}}\,\mathrm{d}r$$

which implies

$$\int_0^{t_0}\frac{\mathrm{d}t}{a(t)}=\int_0^{r_E}\frac{\mathrm{d}r}{\sqrt{1-k\,r^2}}$$

Thus, the largest physical distance d_H we can observe today is (the horizon distance) given by

$$d_H(t_0)=a(t_0)\int_0^{r_E}\frac{\mathrm{d}r}{\sqrt{1-k\,r^2}}=a(t_0)\int_0^{t_0}\frac{\mathrm{d}t}{a(t)}=a(t_0)\int_{\eta(0)}^{\eta(t_0)}\mathrm{d}\eta$$

(Note that $d_H(t_0)$ is simply $\int_0^{r_E}\sqrt{g_{\mathrm{rr}}}\,\mathrm{d}r$.)

d_H sets a limit for what part of the universe is in causal contact.

Redshift

Since the universe is expanding, all lengths are stretched out. In particular this applies to the wave lengths of light. Consider two successive wave crests emitted at t_E and $t_E+\delta t_E$ and observed at t_0 and $t_0+\delta t_0$. We have

$$\int_{t_E}^{t_0}\frac{\mathrm{d}t}{a(t)}=\int_{t_E+\delta t_E}^{t_0+\delta t_0}\frac{\mathrm{d}t}{a(t)}\quad\Leftrightarrow\quad\int_{t_0}^{t_0+\delta t_0}\frac{\mathrm{d}t}{a(t)}=\int_{t_E}^{t_E+\delta t_E}\frac{\mathrm{d}t}{a(t)}$$

For $\delta t\ll t$ and using $\lambda=\delta t$ we get

$$\frac{\lambda_0}{\lambda_E}=\frac{a(t_0)}{a(t_E)}.$$

It is very common to define the redshift z via

$$1+z\equiv\frac{\lambda_0}{\lambda_E}=\frac{a(t_0)}{a(t_E)}$$

Luminosity distance

How does a distant light source look in observations here on earth, and how can it be used to determine the cosmological parameters?

There is no method that we can use to directly obtain distances to cosmological objects. However, if we know the absolute luminosity L (energy/time) of an object (such an object is called a *standard candle*), then we can define its luminosity distance my measuring the flux F (energy per time and area), via

$$d_L^2=\frac{L}{4\pi F}.$$

If there was no expansion, then a telescope with area A would intercept a fraction

$$\frac{A}{4\pi(a(t_E)\,r)^2}$$

of the emitted photons. But because of the expansion only a fraction

$$\frac{A}{4\pi(a(t_0)r)^2}$$

are observed.

The flux has additional factors of $1+z$, since photons are redshifted by $1+z=\frac{a(t_0)}{a(t_E)}$.

Furthermore, the time interval between photon emissions will also be increased by a redshift factor $1+z$.

$$F=\frac{L}{4\pi\, a^2(t_0)\, r^2\, (1+z)^2}\equiv\frac{L}{4\pi\, d_L^2}$$

Eliminating the parameter r one finds

$$d_L=\frac{1}{H_0}\left(z+\frac{1}{2}(1-q_0)z^2+\cdots\right)$$

Angular distance

Another measurement of distance is the angular distance d_A:

$$d_A=\frac{D}{\delta\theta}$$

where D = proper size of the object, assumed to be known (standard ruler). $\delta\theta$ is the angular size of the object.

$$D=a(t_E)\, r\, \delta\theta\Rightarrow d_A=a(t_E)r=\frac{a(t_0)}{1+z}\, r$$

Thus

$$d_A=\frac{d_L}{(1+z)^2}$$