2009–01–21

Lecturer: Niclas Wyllard

Rudiments of general relativity

The fundamental object in Einstein's theory of gravitation is the metric tensor. The distance between two (spacetime) points with infinitesimal separation is:

$$\mathrm{d}s^2 = g_{\mu\nu}(x)\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \stackrel{\text{def}}{=\!=} \sum_{\mu,\nu=0}^{3} g_{\mu\nu}(x)\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$$

The index μ runs over four values, three space indices and one time index.

Examples:

• $\mathbb{R}^2$:

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2, \quad \text{or in polar coordinates:} \quad \mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2$$

• S^2 (the two-dimensional [surface of a] sphere).

$$\mathrm{d}s^2 = R^2\big(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\big)$$

• 4-dimensional Minkowski space:

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 \equiv \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$$

(Note the choice of sign convention, which differs from that used in the course "Gravitation and Cosmology".)

When is space flat/curved?

Can be determined from the Riemann tensor $R^\mu{}_{\nu\rho\sigma}$:

$$R^\mu{}_{\nu\rho\sigma}(x) \equiv 0 \quad \Leftrightarrow \quad \text{space(time) is flat}$$

$$R^\mu{}_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\mu_{\lambda\rho}\,\Gamma^\lambda_{\nu\sigma} - \Gamma^\mu_{\lambda\sigma}\Gamma^\lambda_{\nu\rho}$$

where

$$\Gamma^\mu_{\sigma\delta} = \frac{g^{\mu\lambda}}{2}(\partial_\sigma g_{\lambda\delta} + \partial_\delta\, g_{\lambda\sigma} - \partial_\lambda g_{\delta\sigma})$$

where $g^{\mu\nu} g_{\nu\sigma} = \delta^\mu_\sigma$ (the $g^{\mu\nu}$ is the components of the matrix inverse of the matrix with components $g_{\nu\sigma}$).

Einstein's equations

Define $R_{\mu\nu\rho\sigma} = g_{\mu\lambda} R^\lambda{}_{\nu\rho\sigma}$ and $R_{\mu\nu} = g^{\rho\sigma} R_{\rho\mu\sigma\nu}$. The Ricci tensor is symmetric in its indices: $R_{\mu\nu} = R_{\nu\mu}$. Also define $R = g^{\mu\nu} R_{\mu\nu}$. Using these objects, Einstein's equations read

$$R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} R = 8\pi G\, T_{\mu\nu}$$

where $T_{\mu\nu}$ is the energy-momentum tensor, which acts as a source term. These partial differential equations (10 of them) determine the metric of spacetime.

Standard model of cosmology (FRW model)

The universe is assumed to be homogeneous and isotropic. Homogeneous means that (for a given time) physics is the same at every point. Isotropy means that the physics is the same in every direction from a given point. Homogeneity does not necessarily imply isotropy. However, isotropy at *every* point implies homogeneity.

One can show that homogeneity and isotropy implies the line element (the metric) can be written

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \mathrm{d}t - a^2(t)\left(\frac{\mathrm{d}r^2}{1-k\,r^2} + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\varphi^2\right)$$

where $k \in \{-1, 0, +1\}$.

It is assumed that the "energy" / matter of the universe is a "perfect fluid" described by its energy density ρ, its pressure p and its four-velocity, u^μ. The energy momentum tensor is

$$T_{\mu\nu} = (p+\rho)u_\mu u_\nu - p\,\eta_{\mu\nu}$$

In general there is an equation of state: $p = p(\rho)$. With the above assumptions, Einstein's equations $R_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}\,R = 8\pi G\,T_{\mu\nu}$ then leads to

$$\begin{cases} \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho \\ \frac{2\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = -8\pi G\,p \end{cases}, \qquad \dot{} \equiv \frac{\mathrm{d}}{\mathrm{d}t} \tag{1}$$

Conservation of energy-momentum, $\mathrm{D}_\mu T^{\mu\nu} = 0$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho a^3) = -p\,\frac{\mathrm{d}}{\mathrm{d}t}(a^3)$$

which also follows from the equations (1), and it has a simple interpretation since a^3 can be thought of as a volume, $a^3 \sim V$, and similarly $\rho V \sim E$, the total energy. Now the above simply says $\mathrm{d}E = -p\,\mathrm{d}V$.

Some examples:

- Ultra-relativistic gas (e.g. photons, radiation): $p = \rho/3$.

$$\frac{2\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\,\frac{k}{a^2} = 0$$

Introduce η implicitly by $\mathrm{d}\eta = \mathrm{d}t/a(t)$:

$$a'(\eta) \equiv \frac{\mathrm{d}a}{\mathrm{d}\eta} = \frac{\mathrm{d}a}{\mathrm{d}t}\,\frac{\mathrm{d}t}{\mathrm{d}\eta} = \dot{a}(t)\,a(t)$$

$$a'' = \frac{\mathrm{d}^2 a}{\mathrm{d}\eta^2} = \frac{\mathrm{d}}{\mathrm{d}\eta}(a\,\dot{a}) = \frac{\mathrm{d}}{\mathrm{d}t}(a\,\dot{a})\,a = \ddot{a}\,a^2 + \dot{a}^2\,a$$

$$a'' + k\,a = 0 \quad \Rightarrow \quad a(\eta) = C\cdot\begin{cases} \sinh(\eta), & k = -1 \\ \eta, & k = 0 \\ \sin(\eta), & k = +1 \end{cases}$$

$t = t(\eta)$ can be found from $\mathrm{d}t/\mathrm{d}\eta = a(\eta)$.

Also, $\dot{\rho}\, a^3 + 3\rho\, \dot{a}\, a^2 + \rho\, a^2 \dot{a} = 0$

$$\Rightarrow \frac{\dot{\rho}}{\rho} + 4\,\frac{\dot{a}}{a} = 0 \quad \Rightarrow \quad \boxed{\rho \propto a^{-4}}$$

• Nonrelativistic matter (e.g. non-relativistic neutrons, protons, etc.). $p = 0$.

$$a'' + k\, a = \frac{4\pi G}{3}\, \rho\, a^3$$

and

$$\frac{\dot{\rho}}{\rho} + \frac{3\dot{a}}{a} \quad \Rightarrow \quad \boxed{\rho \propto a^{-3}}$$

$$a(\eta) = C \cdot \begin{cases} \cosh \eta - 1, & k = -1 \\ \eta^2, & k = 0 \\ 1 - \cos \eta, & k = +1 \end{cases}$$

• Vacuum energy (e.g. cosmological constant). $p = -\rho$. $\dot{\rho} = 0$, i.e.

$$\rho = \text{const} \equiv \frac{\Lambda}{8\pi G}$$

$$(8\pi G\, T_{\mu\nu} = \Lambda\, g_{\mu\nu})$$

$$2\,\frac{\ddot{a}}{a} = 8\pi G\Big(-\frac{\rho}{3} - p \Big) = \frac{2}{3}\,\Lambda$$

$$\Leftrightarrow \quad \ddot{a} - \frac{\Lambda}{3}\, a = 0$$

$$a(t) = \sqrt{\frac{3}{\Lambda}} \begin{cases} \sinh\Big(\sqrt{\frac{\Lambda}{3}}\, t \Big), & k = -1 \\ \exp\Big(\sqrt{\frac{\Lambda}{3}}\, t \Big), & k = 0 \\ \cosh\Big(\sqrt{\frac{\Lambda}{3}}\, t \Big), & k = +1 \end{cases}$$

Newtonian (non-relativistic) analysis

Consider an infinite, expanding homogeneous and isotropic universe filled with "dust", i.e. matter whose pressure p is negligible compared to its (energy) density ρ.

Consider an arbitrary sphere with radius R. Because of the expansion

$$R(t) = \frac{a(t)}{a_0}\, R_0$$

Assumption: The effect of the matter outside of the sphere on a particle inside the sphere is zero. (Birkhoff's theorem in GR [What?! Looking up Birkhoff's theorem, I see no connection.]).

Total mass inside the sphere is conserved, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t} M = 0.$$

Since

$$M = \rho(t)\,\frac{4\pi}{3}\,R^3(t) = \rho(t)\left(\frac{a(t)}{a_0}\right)^3 \cdot \frac{4\pi R_0^3}{3}$$

we get

$$\dot{\rho} = -\,\frac{3\dot{a}}{a}\,\rho$$

Equation of motion for a particle on the surface of the sphere:

$$m\,\ddot{R}(t) = -\,\frac{G\,m\,M}{R^2}$$

Using $R(t) = \frac{a(t)}{a_0}\,R_0$ we get

$$\ddot{a} = -\,\frac{4\pi G}{3}\,\rho\,a.$$

Agrees with previous expressions! (when $p = 0$). Reason: the sphere can be made arbitrarily small.