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Last time, we found for a free particle:

$$\Psi(x,t) = A\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}R}, \quad \text{where } R = \int_0^t \mathrm{d}t'\,L(x,\dot{x}), \quad \dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t}$$

$$\psi(x) = A\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}S}, \quad \text{where } S = \int_{x_{\text{init}}}^{x_{\text{final}}} \mathrm{d}x'\,p(x')$$

S and R are classical actions. The expressions for S and R are valid in a general setting. Variational principles: $\delta R = 0$ for a given t, $\delta S = 0$ for given energy. These variational principles give the classical path.

Let us now consider a particle bound in a potential, still in one dimension.

$$\left[-\frac{\hbar}{2\,m}\frac{\partial^2}{\partial x^2} + V(x)\right]\psi(x) = E\,\psi(x)$$

First, we want to rewrite this so that the classical momentum $p(x)$ appears.

$$H = \frac{p^2}{2\,m} + V(x) = E, \quad p \to \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial x}$$

$$p(x) = \sqrt{2\,m\,(E - V(x))}$$

$$\hbar^2\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + p^2(x)\,\psi(x) = 0$$

Ansatz:

$$\psi(x) = A\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S(x)}$$

What will the action $S(x)$ be?

$$\frac{\mathrm{d}\psi}{\mathrm{d}x} = \frac{\mathrm{i}}{\hbar}S'(x)\,\psi(x)$$

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} = -\frac{[S'(x)]^2}{\hbar^2}\,\psi(x) + \frac{\mathrm{i}}{\hbar}S''(x)\,\psi(x) = \left[-\frac{[S'(x)]^2}{\hbar^2} + \frac{\mathrm{i}}{\hbar}S''(x)\right]\psi(x)$$

Looks complicated. We can't just look at this and see if S is a classical action. When $\hbar \to 0$ we expect classical results. So we expand S in powers of $\hbar$:

$$S(x) = S_0(x) + \hbar\,S_1(x) + \hbar^2 S_2(x) + \mathcal{O}(\hbar^3)$$

S_1 and S_2 will be just quantum mechanical corrections to S_0. We find:

$$\left[-\left[S_0'(x)\right]^2 + p^2(x) + \left(-2\,S_0'\,S_1' + \mathrm{i}\,S_0''\right)\hbar + \left(-2\,S_0'S_2' - \left(S_1'\right)^2 + \mathrm{i}S_1''\right)\hbar^2 + \mathcal{O}(\hbar^3)\right]\psi(x) = 0$$

To order $\hbar^0$:

$$S_0' = \pm\, p(x)$$

$$S_0 = \pm \int^x \mathrm{d}x'\, p(x')$$

This is just the classical action, the Maupertuis action.

To order $\hbar^1$:

$$S_1' = \frac{\mathrm{i}\, S_0''}{2\, S_0'} = \frac{\mathrm{i}}{2} \frac{\mathrm{d}}{\mathrm{d}x} \log |S_0'|$$

$$S_1(x) = \frac{\mathrm{i}}{2} \log |S_0'(x)| = \frac{\mathrm{i}}{2} \log |p(x)|$$

If you want an additive constant there, just add it in. It will become a normalisation factor in the end:

$$\psi(x) = \frac{A}{\sqrt{|p(x)|}} \exp\!\left(\pm \frac{\mathrm{i}}{\hbar} \int^x \mathrm{d}x'\, p(x') \right)$$

WKB wavefunction.

Borel-resummation

$$f(x) = \sum_l c_l x^l$$

$$l! = \int_0^\infty \mathrm{d}t\, t^l\, \mathrm{e}^{-t}$$

$$B(x) = \sum_l \frac{c_l}{l!} x^l$$

$$f(x) = \sum_l \frac{c_l}{l!} x^l \int_0^\infty \mathrm{d}t\, t^l\, \mathrm{e}^{-t} \overset{*}{=} \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-t} \sum_l \frac{(x\, t)^l}{l!} c_l = \int_0^\infty \mathrm{d}t\, B(x\, t)$$

* Not really a legitimate step, mathematically.

Counterexample: $f(x) = \exp(-1/x)$. Expanding about $x = 0$ gives us $c_l = 0\ \forall l$.

We think of $V(x)$ as the potential for the harmonic oscillator in these examples.

$E < V(x)$:

$$\psi(x) = \frac{1}{\sqrt{|p(x)|}} \left(\mathrm{e}^{\frac{1}{\hbar} \int^x \cancel{\mathrm{d}x'} |p(x')|} + \mathrm{e}^{-\frac{1}{\hbar} \int^x \mathrm{d}x' |p(x')|} \right)$$

Exponentially growing solution is unphysical. Note that there is an undetermined constant hidden in the indefinite integral.

$E > V(x)$:

$$\psi(x) = \frac{1}{\sqrt{p(x)}}\left[A_+ \, \mathrm{e}^{+\frac{\mathrm{i}}{\hbar}\int_{x_0}^{x} \mathrm{d}x' \, p(x')} + A_- \, \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\int_{x_0}^{x} \mathrm{d}x' \, p(x')}\right]$$

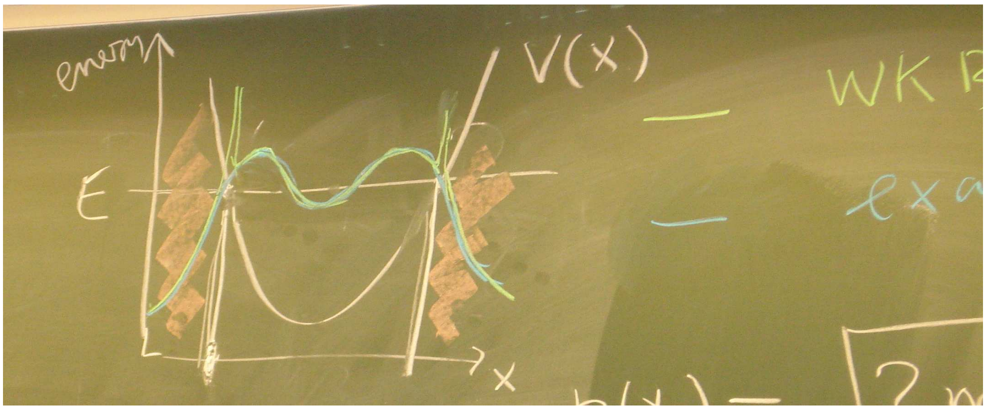

Figure 1. There is a problem at $p(x) = 0$. We have turning points. There, we will find that the quantum mechanical phase makes a jump. The green curve is the WKB approximation, which is close to the exact solution (blue) away from the turning points. The shaded red area is the classically forbidden region.

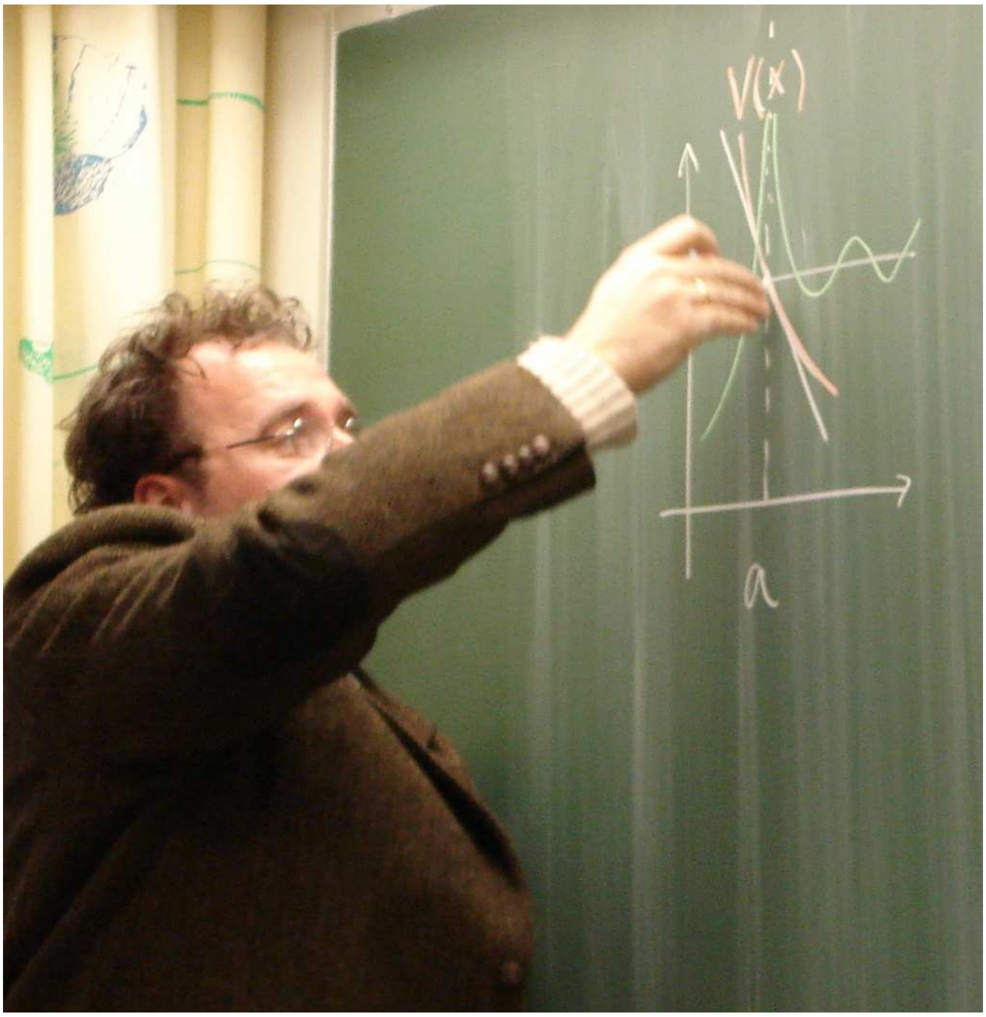

Figure 2. We approximate $V(x)$ at the turning point with its linearisation...

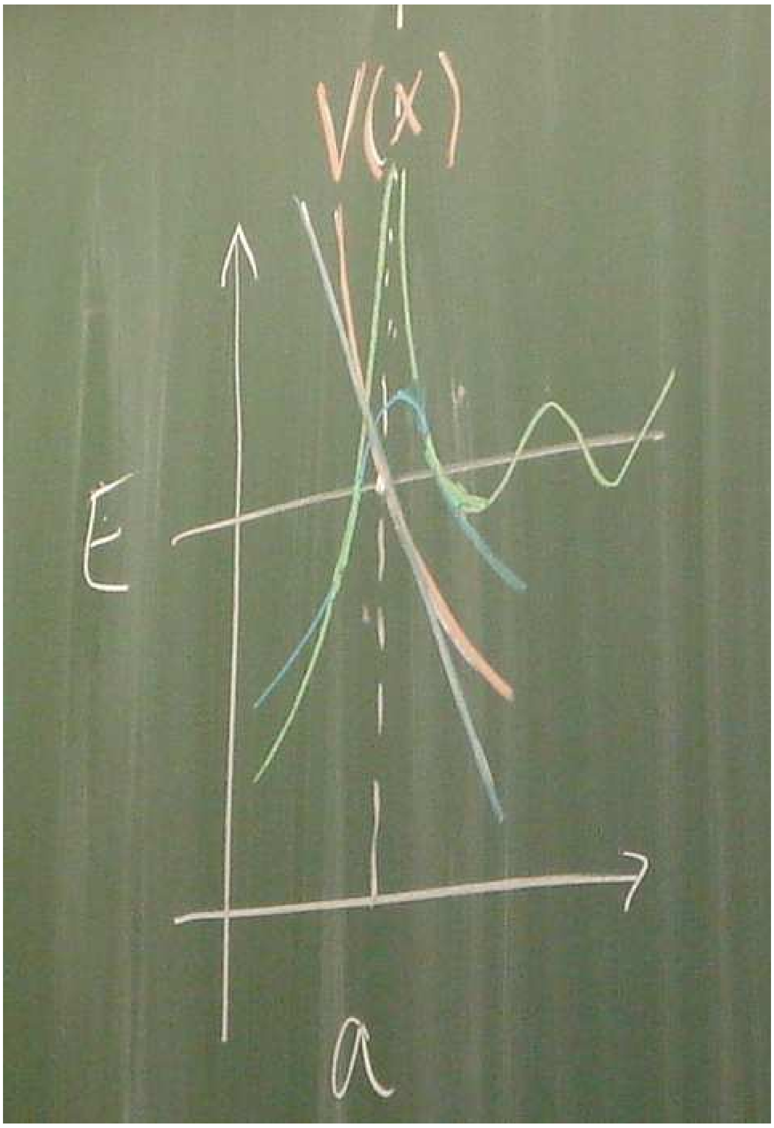

Figure 3. ... and calculate an exact wavefunction (blue) for the linearised potential, and use this intermediate wavefunction to patch the WKB approximations (green) in the different regions.

$$V(x) = E - F \cdot (x - a)$$

1. Classical motion in the vicinity of the turning point a.

$$p(x) = \pm\sqrt{2\,m\,(E - V(x))} = \pm\sqrt{2\,m\,F \cdot (x - a)}$$

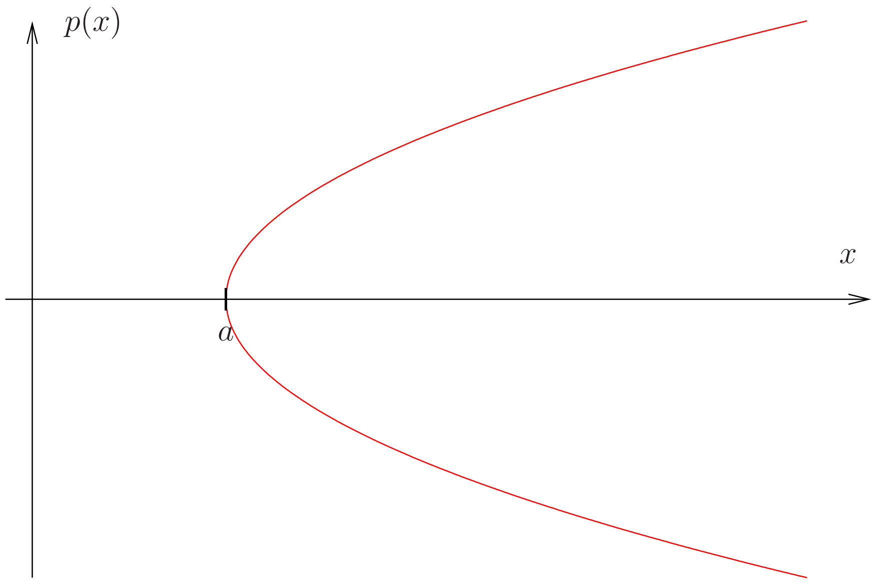

Figure 4. $p(x)$ plot.

$$S_0(x) = \pm \int_a^x \mathrm{d}x'\, p(x') = \pm \frac{2}{3}\,(2\,m\,F)^{1/2} \cdot (x-a)^{3/2}$$

2. Exact solution of the linearised problem.

$$\left[\frac{\hat{p}^2}{2\,m} - F\,(x-a)\right]\psi(x) = a$$

Going to momentum representation:

$$\left[\frac{p^2}{2\,m} - F\,(\hat{x}-a)\right]\varphi(p) = 0, \quad \hat{x} = \mathrm{i}\,\hbar\,\frac{\partial}{\partial p}$$

$$\psi(x) = \int_{-\infty}^{\infty} \mathrm{d}p\,\varphi(p)\,\mathrm{e}^{\mathrm{i}px/\hbar}$$

$$\varphi(p) = C\,\exp\!\left(-\frac{\mathrm{i}}{\hbar}\left(\frac{p^3}{6\,m\,F} + a\,p\right)\right)$$

$$\psi(x) = C\int_{-\infty}^{\infty} \mathrm{d}p\,\exp\!\left(-\frac{\mathrm{i}}{\hbar}\left(\frac{p^3}{6\,m\,F} + (a-x)\,p\right)\right)$$