## 2008–11–26

$$\langle c_{r\uparrow}^{\dagger} c_{r\downarrow}^{\dagger} \rangle \sim \Phi(r)$$

$$\langle c_{p\uparrow}^{\dagger} c_{-p\downarrow}^{\dagger} \rangle = \Phi_p$$

$$|\psi_G\rangle = \prod_{k<k_F} c_{k\uparrow}^{\dagger} c_{-k\downarrow}^{\dagger} |0\rangle$$

$$|\psi_G\rangle = \prod_k \left( u_k + v_k c_{k\uparrow}^{\dagger} c_{-k\downarrow}^{\dagger} \right) |0\rangle, \quad u_k^2 + v_k^2 = 1$$

**Figure 1.** Ordinary Fermi gas

Self-consistent equation at finite temperature

$$-\frac{\Delta}{U_0} = \frac{1}{2} \sum_p \frac{\Delta}{E_p} \tanh \frac{\beta E_p}{2}$$

$$E_p = \sqrt{\varepsilon_p^2 + \Delta^2}$$

BCS self-consistent equations.

**Figure 2.** Plot of $\Delta$.

$$\begin{pmatrix} \varepsilon_p & \Delta \\ \Delta & -\varepsilon_p \end{pmatrix} \begin{pmatrix} u_k \\ v_k \end{pmatrix} = E_p \begin{pmatrix} u_k \\ v_k \end{pmatrix}, \quad u_k^2 + v_k^2 = 1$$

$$H = \sum_p \varepsilon_p c_{p\sigma}^{\dagger} c_{p\sigma} + \sum_p \Delta_p c_{p\uparrow}^{\dagger} c_{-p\downarrow}^{\dagger} + \text{complex conjugate}$$

$$\Delta p = \sum_q \langle c_{q\uparrow}^{\dagger} c_{-q\downarrow}^{\dagger} \rangle U_{p-q}$$

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### Broken Symmetries, Conservation Laws

Noether's theorem. For every continuous symmetry there is a conservation law. Conservation law: $[H, Q_\mu] = 0$, $Q_\mu$ is conserved.

$$\frac{\mathrm{d}Q_\mu}{\mathrm{d}t} \propto [Q_\mu, H] = 0.$$

Symmetry operator:

$$S = \mathrm{e}^{\mathrm{i}\lambda_\mu Q_\mu} = \text{group of symmetry transformations}$$

Say $Q = \text{charge}$.

$$Q = \sum_r a_r^\dagger a_r$$

$$S = \exp\left( \mathrm{i}\,\lambda \sum_r a_r^\dagger a_r \right)$$

$$S^\dagger a_r^\dagger S = \exp\left( \mathrm{i}\,\lambda \sum_{r'} a_{r'}^\dagger a_{r'} \right) a_r^\dagger \exp\left( -\mathrm{i}\,\lambda \sum_{r'} a_{r'}^\dagger a_{r'} \right) =$$

$$= \exp\left( \mathrm{i}\lambda\, a_r^\dagger a_r \right) a_r^\dagger \exp(-\mathrm{i}\lambda a_r^\dagger a_r) = \exp(\mathrm{i}\lambda)\, a_r^\dagger$$

Charge operator

$$Q = \sum_r a_r^\dagger\, a_r$$

becomes the symmetry operator $\mathrm{e}^{\mathrm{i}\Phi Q}$.

Helium

$$E = \frac{1}{2} \int \left( \kappa |\nabla\Phi|^2 + \mu\, \Phi^2 + \lambda |\Phi|^4 \right) \mathrm{d}^d r$$

$\Phi$ was a field that we associated with the condensate in the $k = 0$ state. Assume $\Phi = \text{const}$.

$$E = \left( \mu |\Phi|^2 + \lambda |\Phi|^4 \right) \Omega$$

$\Phi = 0$ minimum solution. $\langle b_r^\dagger \rangle = 0$.

**Figure 3.** Instability when $\mu$ crosses zero.

$\langle b_r^\dagger \rangle \neq 0 \Rightarrow$ condensation.

**Figure 4.** $\Phi_R$ is the real part, $\Phi_I$ is the imaginary part.

How many degrees of freedom are there in the complex field? Two — one real, and one imaginary.

2 massive scalar fields in the non-condensed state (gammetic state). $\Phi = 0$.

Symmetry breaking. $\Phi = |\Phi_R| + \delta\Phi_I + \delta\Phi_R$. One massive field and one massless. Goldstone theorem. Symmetry breaking of a global symmetry $\Rightarrow$ massless field.

Superconductivity

$$E = \frac{1}{2} \int \left( \kappa |\nabla\Phi|^2 + \mu\, \Phi^2 + \lambda |\Phi|^4 \right) \mathrm{d}^d r \quad ??$$

$$\Phi(r) = \langle c_{r\uparrow}^\dagger c_{-r\downarrow}^\dagger \rangle$$

$$\langle c_{r\uparrow}^\dagger c_{r'\downarrow}^\dagger \rangle \langle c_{r\downarrow} c_{r'\uparrow} \rangle$$

$$\Phi(r) \rightarrow \exp\left( -2\,\mathrm{i}\, \frac{e}{c}\, \Lambda(r,t) \right) \Phi(r)$$

gauge: $c_{r\sigma}^{\dagger} \rightarrow e^{ie/c} c_{r\sigma}^{\dagger}$

$$\nabla\Phi \rightarrow \left(\nabla\Phi - 2\,\mathrm{i}\,\frac{e}{c}\,\nabla\Lambda\right)\Phi(r)$$

$$E = \int \kappa \left|\left(\nabla - 2\,\mathrm{i}\,\frac{e}{c}\boldsymbol{A}\right)\Phi\right|^2 + \mu|\Phi|^2 + \lambda|\Phi|^4 + \int \frac{1}{8\pi}h^2\,\mathrm{d}^3r \quad \text{where } h = \nabla\times\boldsymbol{A}$$

The gauge field has a life of its own, and it has an energy. Even in the absence of a superconductor.

$$\int \frac{1}{8\pi}h^2\,\mathrm{d}^3r = \frac{1}{8\pi}\int \underbrace{(\varepsilon_{ijk}\,\partial_j\,A - k)}_{h_i}\,h_i = -\frac{1}{8\pi}\int A_k\varepsilon_{ijk}\partial_j h_i = -\frac{1}{8\pi}\int A\cdot(\nabla\times h)$$

$$0 = \frac{\delta E}{\delta\Phi^*} = -\kappa\left(\nabla - \frac{2\,\mathrm{i}\,e}{c}\boldsymbol{A}\right)^2\Phi + \mu\Phi + \lambda|\Phi|^2\,\Phi$$

$$0 = \frac{\delta E}{\delta A} = \frac{1}{4\pi}\nabla\times h - \frac{2\,e\,\kappa}{c}\left[\Phi^*\,\mathrm{i}\left(\nabla - \frac{2\,\mathrm{i}\not{e}}{\not{c}}A\right)\Phi + \text{complex conjugate}\right] =$$

$$= \frac{1}{4\pi}\nabla\times h - \frac{e\mathrm{i}}{m\,c}(\Phi^*\nabla\Phi - \Phi\,\nabla\Phi^*) + \frac{4e^2\kappa}{c^2}\,|\Phi|^2\,A$$

$$j = \frac{c}{4\pi}\,\nabla\times h$$

$$j = \frac{e}{m}\,\mathrm{Im}(\Phi^*\nabla\Phi - (\nabla\Phi^*)\Phi) + \frac{4\,e^2\kappa}{c}|\Phi^2|\,\boldsymbol{A}$$

$$\boldsymbol{A} \rightarrow \boldsymbol{A} + \nabla\Lambda$$

Use the gauge freedom in $\boldsymbol{A}$ to get rid of the phase of $\Phi$. Choose $\Phi$ real.

$$\boldsymbol{j} = \frac{4\,e^2\kappa}{c}\,\boldsymbol{A}\,|\Phi|^2$$

$$-\frac{c}{4\pi}\nabla\times(\nabla\times A) = \frac{4e^2}{c}\,k\,|\Phi^2|\,\boldsymbol{A}$$

$$\nabla^2 A = m^2 A$$

where

$$m^2 = \frac{4\,e^2\,\kappa}{c}\cdot\frac{c}{4\pi}\,|\Phi^2|$$

$$\nabla^2\boldsymbol{A} = 0 \quad \text{(ordinary free space)}$$

$$\Box^2\boldsymbol{A} = 0$$

$$\left(\nabla^2 - \frac{\partial^2}{\partial t^2}\right)A = m^2\,\boldsymbol{A}$$

This is the Higgs mechanism.

$|\Phi|$. 2 massive scalar fields. $\boldsymbol{A}$: 2 massless gauge fields. Gauge symmetry breaking. 1 massive gauge field. “massive photon”. 3 massive degrees of freedom. $\Phi$ real. 1 massive degree of freedom left.