2008–11–25

$$2\pi \int \left(a\, r^2 + b\, r + c + \frac{d}{r} \right) \mathrm{e}^{-r/l}\, \mathrm{e}^{\mathrm{i} k_z r_z \cos\theta}\; r^2 \sin\theta\, \mathrm{d}r\, \mathrm{d}\theta$$

Wick's theorem, a.k.a. Cumulent expansion.

$f^\dagger = (a_1^\dagger, \ldots, a_k^\dagger, a_1, \ldots, a_k)$.

$$\langle f_i f_j \rangle = \sum_{p} (\xi)^p \, \langle f_{p_1} f_{p_2} \rangle \cdots \langle f_{p_{k-1}} f_{p_k} \rangle$$

sum over distinct ordered pairs.

Example: $\langle a_1^\dagger a_1 \rangle = \langle a_1^\dagger a \rangle$

Example: $\langle a_1^\dagger a_2^\dagger a_2\, a_1 \rangle = \langle a_1^\dagger a_1 \rangle \langle a_2^\dagger a_2 \rangle - \langle a_1^\dagger a_2 \rangle \langle a_2^\dagger a_1 \rangle + \langle a_1^\dagger a_2^\dagger \rangle \langle a_2\, a_1 \rangle$.

Now, let's get started with the weakly interacting Fermi gas. The Hamiltonian we are going to be using is the good old second quantised Hamiltonian:

$$H = \sum_{p\sigma} \varepsilon_p\, c_{p\sigma}^\dagger\, c_{p\sigma} + \sum_{\substack{kpq\\ \sigma\sigma'}} U_k\, c_{p+k,\sigma}^\dagger\, c_{q-k,\sigma'}^\dagger\, c_{q,\sigma'}\, c_{p,\sigma}$$

σ is spin index, $\uparrow$ or $\downarrow$.

$U(r)$. $n(r)(n(r')-1)$. $n(r)\, n(r') = : c_{r\sigma}^\dagger c_{r\sigma} c_{r'\sigma'}^\dagger c_{r'\sigma'} :$. Thus, this is the Hamiltonian with spin:

$$H = \sum_{p\sigma} \varepsilon_p\, c_{p\sigma}^\dagger\, c_{p\sigma} + \sum_{\substack{kpq\\ \sigma\sigma'}} U_k\, c_{p+k,\sigma}^\dagger\, c_{q-k,\sigma'}^\dagger\, c_{q,\sigma'}\, c_{p,\sigma}$$

Ground state $\Rightarrow$ Fermi surface.

Figure 1.

This is what the ground state looks like in the absence of an interaction. Let us take a look at what an interaction can do. The interaction part of the Hamiltonian punches two holes in the ground state.

$$|\psi_G\rangle = \prod_{k,\sigma_k}^{\varepsilon_k < \varepsilon_T} c_{k\sigma}^\dagger |0\rangle$$

If I add a creation operator that is already present in this list, I get 0. So the only way to get something from the interaction part, is if we create new particles outside the Fermi surface.

Figure 2.

The ground state is disturbed, and the electrons near the Fermi surface do something. For reasonably small k we are near the Fermi surface.

The potential as it was written here, is just as intractable as the boson case. It took until 1960 before someone realized what to do with this. It took 30 years to develop methods to deal with this.

The Colomb potential is repulsive. An effective attractive potential is induced in a lattice. [A lot of handwaving.] You can get excitations of negative energy. The Fermi surface is unstable.

$c^\dagger_{r\sigma} c^\dagger_{r'\sigma'} c_{r'\sigma'} c_{r\sigma}$.

Grouping. Hartree approximation. $\underline{.\quad ,—, \quad .}$ No superconductivity.

Fock approximation. $\underline{.\quad ,—.}—,$

Anomalous $\langle c^\dagger_{r\sigma} c^\dagger_{r'\sigma'}\rangle\langle c_{r'\sigma'} c_{r\sigma}\rangle$.

$\langle c^\dagger_{r\sigma} c^\dagger_{r'\sigma'}\rangle$ Under gauge transformations, what happens? $c^\dagger_{r\sigma} \to \mathrm{e}^{\mathrm{i}\varphi} c^\dagger_{r\sigma}$. This expectation value violates gauge invariance. It does not allow a definite number of particles.

$$\Delta = \langle c^\dagger_{p+k,\uparrow} c^\dagger_{q-k,\downarrow}\rangle$$

$\sigma = -\sigma' \Rightarrow$ no spin polarization. $p + k = -(q-k) \Rightarrow$ translational invariance.

$$\Delta_p = \langle c^\dagger_{p\uparrow} c^\dagger_{-p\downarrow}\rangle \neq 0$$

$$H \approx \sum_p \varepsilon_p c^\dagger_{p\sigma} c_{p\sigma} + \sum_{pq} U_{p-q} \left[\underbrace{\left(c^\dagger_{p+k,\sigma} c^\dagger_{q-k,\sigma} - \langle c^\dagger c^\dagger\rangle \right)}_{\text{small}} + \langle c^\dagger c^\dagger\rangle \right] \times \left[\underbrace{\left(c_{q\sigma'} c_{p\sigma} - \langle cc\rangle \right)}_{\text{small}} + \langle cc\rangle \right]$$

$$H \approx \sum_p \varepsilon_p c^\dagger_{p\sigma} c_{p\sigma} + \sum_{pqk} \left(-\langle c^\dagger c^\dagger\rangle\langle cc\rangle \right) + \sum_{pq} U_{p-q} \left(\langle c^\dagger_{p\uparrow} c^\dagger_{-p\downarrow}\rangle c_{q\downarrow} c_{-q\uparrow} + \langle c_{q\downarrow} c_{-q\uparrow}\rangle \, c^\dagger_{p\uparrow} c^\dagger_{p\downarrow} \right)$$

This is called the BCS Hamiltonian.

$$H_{\mathrm{BCS}} = \sum_p \varepsilon_p c^\dagger_{p\sigma} c_{p\sigma} + \sum_p \left(\Delta p \, c^\dagger_{p\uparrow} \, c^\dagger_{-p\downarrow} + \mathrm{cc} \right) + \text{const}, \quad \Delta p = \sum_q \langle c^\dagger_{q\uparrow} c^\dagger_{-q\downarrow}\rangle U_{p-q}$$

$\varepsilon_p = p^2 2m - \mu$ with chemical poetential.

$$H = \sum_p \begin{array}{c|cc|} & c_{p\uparrow} & c^\dagger_{-p\downarrow} \\ c^\dagger_{p\uparrow} & \varepsilon_p & \Delta p \\ c_{-p\downarrow} & \Delta p & -\varepsilon_p \end{array} = \sum_p f^\dagger_\alpha h_{\alpha\beta} f_\beta$$

Energy eigenvalues.

$$(\varepsilon_p - E_p)(-\varepsilon_p - E_p) = \Delta p^2$$

$$E_p^2 = \Delta p^2 + \varepsilon_p^2$$

$$E_p = \pm\sqrt{\varepsilon_p^2 + \Delta p^2}$$

Figure 3. Gap at the Fermi surface

$$c_{p\uparrow} = u_p\,\hat{c}_{p\uparrow}^{\dagger} + v_p\,c_{-p\downarrow}$$

$$c_{-p\downarrow} = -v_p\,c_p^{\dagger} + u_p\,c_{-p\downarrow}$$

$$h = U^{\dagger}\,h\,U, \quad U = \begin{pmatrix} u & v \\ -v & u \end{pmatrix}$$

$$(u, v)\colon \quad \begin{pmatrix} \varepsilon_p - E_p & \Delta p \\ \Delta p & -\varepsilon_p - E_p \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = 0$$

$$(u, v) = \frac{(\Delta_p, E_p - \varepsilon_p)}{\sqrt{(E_p - \varepsilon_p)^2 + \Delta p^2}}$$

$$\langle c_{p\uparrow}^{\dagger} c_{-p\downarrow} \rangle = \langle -\hat{c}_{p\uparrow}\,u_p c_{p\uparrow}\,v_p + \underbrace{\ldots}_{\substack{\text{gives}\\ \text{zero}}} \rangle = -u_p v_p \langle \hat{c}_{p\uparrow}^{\dagger} \hat{c}_{p\uparrow} \rangle = -u_p v_p = \frac{-\Delta p (E_p - \varepsilon_p)}{(E_p - \varepsilon_p)^2 + \Delta p^2} =$$

$$= \frac{-\Delta p (E_p - \varepsilon_p)}{E_p^2 - 2\,\varepsilon_p\,E_p + \Delta p^2 + \varepsilon_{p^2}} = \frac{-\Delta p (E_p - \varepsilon_p)}{E_p (E_p - \varepsilon_p)}$$

$$\langle c_{p\uparrow}^{\dagger} c_{-p\downarrow}^{\dagger} \rangle = -\frac{\Delta p}{E_p}$$

$$\Delta p = -\sum_q\,(U_{p-q})\,\frac{\Delta q}{2 E_q}$$

BCS self-consistent equation.

$$E_q = \sqrt{\varepsilon_p^2 + \Delta p^2}$$

Assume scattering with isa δ-function. $U_p = U_0$.

$$\Delta = \sum_q -\frac{U_0 \Delta}{2\sqrt{\varepsilon_p^2 + \Delta^2}}$$

$$\frac{1}{U_0} = -\sum_{\varepsilon} \frac{1}{2\,\sqrt{\varepsilon_p^2 + \Delta^2}}$$

Tinkham, introduction to superconductivity.