2008–11–21

$$\varepsilon_p = \frac{p^2}{2\,m}, \quad n_\varepsilon = \frac{1}{\mathrm{e}^{\beta(\varepsilon-\mu)}+1}$$

As I go to zero temperature, the occupied states above $\varepsilon = 0$ have to go to something finite, ... In dimensions two and below, there is no Bose condensation.

If we have a spectrum that looks like this:

$$\varepsilon_p = |p|^{2+\eta}\cdot\gamma, \quad D \text{ dimensions.}$$

$$\left(A\,r^2 + B\,r + \frac{C}{r}\right)\mathrm{e}^{-\gamma r} \text{ potential}$$

m free parameter, start with mass of Helium. Roton.

That will be the problems due on Wednesday, and they will show up on the web page sometime this afternoon.

$$H = \sum_p \varepsilon_p\, b_p^\dagger\, b_p + \frac{1}{2\,\Omega}\sum_{pqk} b_{p+k}^\dagger\, b_{q-k}^\dagger\, b_q\, b_p\, U_k$$

10 000 molecules is as much as one can simulate classically. Quantum mechanically the situation gets ridiculous. Even for one particle the Hilbert space is infinitely dimensional. Discretising the Hilbert space 1000 × 1000 grid: 10^9 points for one point, $(10^9)^3$ for three. 10^{27} dimensional problem for three particles, and we want the limit with a lot of particles (10^{23} or so). We have to convert the unsolvable problem into a solvable problem without destroying the physics.

$$b_0^\dagger \sim \sqrt{N_0}\cdot\mathrm{e}^{\mathrm{i}\varphi}$$

$$b_r^\dagger = \sum_p \langle r|p\rangle\, b_p^\dagger = \sqrt{\frac{N_0}{\Omega}}\,\mathrm{e}^{\mathrm{i}\varphi} + \sum_p\!\!\!\!/\;\langle r|p\rangle\, b_p^\dagger$$

$$b_r^\dagger = \underbrace{\sqrt{n_0(r)}\,\mathrm{e}^{\mathrm{i}\varphi(r)}}_{\psi^*(r)} + \sum_p\!\!\!\!/\;\langle r|p\rangle\, b_p^\dagger$$

$$\langle H\rangle = -\frac{\hbar^2}{2m}\sum_r \langle\psi|\, b_r^\dagger \frac{\partial^2}{\partial r^2}\, b_r|\psi\rangle + \sum \frac{1}{2}\langle G|b_r^\dagger\, b_{r'}^\dagger b_{r'} b_r|G\rangle$$

$$= -\frac{\hbar^2}{2m}\int \mathrm{d}^3r\,\psi^*(r)\,\frac{\partial^2}{\partial r^2}\psi(r) + \iint \frac{1}{2}|\psi^2(r)|\,\psi^2(r')\,V(r-r') + \ldots$$

$$= \frac{\hbar^2}{2\,m}\int \mathrm{d}^3r\,|\nabla\psi|^2 + \frac{1}{2}\iint \mathrm{d}^3v\,U_0|\psi^4(r)|$$

Classical field theory.

Landau-Ginzburg theory. 2-fluid model.

$\psi \rightarrow \mathrm{e}^{\mathrm{i}\varphi}\psi$ does not change the problem. Global phase transformation, which is a symmetry. The ground state breaks the symmetry. Goldstone: Broken continuous symmetry leads to a massless field. (This is jargon from the particle physics community.)

$$E = \sqrt{p^2 + m^2}$$

Superfluid. What do we mean by momentum? Galilean transformation:

$$\varepsilon \rightarrow \varepsilon - \boldsymbol{p} \cdot \boldsymbol{v}$$

$$\boldsymbol{P} = \int \boldsymbol{p} \cdot n(\varepsilon - \boldsymbol{p} \cdot \boldsymbol{v})\, \mathrm{d}^3 p = - \int \boldsymbol{p}\, (\boldsymbol{p} \cdot \boldsymbol{v})\, \frac{\mathrm{d}n}{\mathrm{d}\varepsilon}\, \mathrm{d}^3 p$$

$$\boldsymbol{P} \cdot \boldsymbol{v} = - \int\ (\boldsymbol{p} \cdot \boldsymbol{v})^2 \frac{\mathrm{d}n}{\mathrm{d}\varepsilon}\, \mathrm{d}^3 p = - \frac{1}{3}\, v^2 \int p^2 \frac{\mathrm{d}n}{\mathrm{d}\varepsilon}\, \mathrm{d}^3 p = - \frac{4\pi}{3} v^2 \int p^4 \frac{\mathrm{d}n}{\mathrm{d}\varepsilon}\, \mathrm{d}p =$$

$$= - \frac{4\pi}{3}\, v^2 \int p^4\, \frac{\mathrm{d}n}{\mathrm{d}p}\, \frac{\mathrm{d}p}{\mathrm{d}\varepsilon}\, \mathrm{d}p = - \frac{4\pi}{3\,C}\, v^2 \int p^4\, \frac{\mathrm{d}n}{\mathrm{d}p}\, \mathrm{d}p = - \frac{4\pi}{3\,C}\, v^2 \int p^4 \mathrm{d}n$$

$$= \frac{4\pi}{3C}\, v^2 \int n(p)\ \mathrm{d}p^4 = \frac{16\pi}{3C}\, v^2 \int p^3\, n(p)\, \mathrm{d}p$$

The surface term: $p^4\ n\big|_0^\infty$

$$\boldsymbol{P} \cdot \boldsymbol{v} = \frac{16\pi}{3}\, \frac{v^2}{c}$$

$$\boldsymbol{P} = \frac{16\pi}{3\,C}\, v \int p^3\, n(p)\, \mathrm{d}p$$

$$= \frac{16\pi}{3C}\, v\, \frac{1}{4\pi} \int p\, n(p)\, \left(p^2 \mathrm{d}p\, 4\pi\right)$$

$$\boldsymbol{P} = v\, \frac{4}{3} \cdot \frac{1}{C} \int p\, n(p)\, \mathrm{d}^3 p$$

$$p = \frac{\varepsilon(p)}{C}$$

$$\boldsymbol{P} = v\, \frac{4}{3}\, \frac{1}{C^2} \int \varepsilon(p)\, n(p)\, \mathrm{d}^3 p = v\, \underbrace{\frac{4}{3}\, \frac{1}{c^2} (E_{\mathrm{tot}})_{\mathrm{quasiparticles}}}_{\text{inertial mass}}$$

Wicks Theorem. Also called the Cumulent expansion.

$$f_\alpha^\dagger = \left(a_1^\dagger, \ldots, a_n^\dagger, a_1, \ldots, a_n\right)$$

If H is a quadratic function of $f_\alpha^\dagger f_\beta$, then

$$\langle f_1 f_2 \ldots f_k \rangle = \sum_p\ (\xi)^p \langle f_{p_1} f_{p_2} \rangle \langle f_{p_3} f_{p_4} \rangle \cdots \langle f_{p_{k-1}} f_{p_k} \rangle$$

Proof in special case:

$$H = \sum_p \varepsilon_p \, a_p^\dagger a_p$$

$$\langle a_k^\dagger a_k \rangle = \frac{1}{Z} \operatorname{tr}\Big(\mathrm{e}^{-\beta H} a_k^\dagger a_k \Big) = \frac{1}{Z} \operatorname{tr}\Bigg(\mathrm{e}^{-\beta\left(\varepsilon_1 a_1^\dagger a_1 + \varepsilon_2 a_2^\dagger a_2 + \cdots \right)} a_k^\dagger a_k \Bigg) =$$

$$= \frac{\operatorname{tr} \; \prod_n \; \mathrm{e}^{-\beta \varepsilon_n a_n^\dagger a_n} \, a_k^\dagger a_k}{\prod_k \; \operatorname{tr} \mathrm{e}^{-\beta \varepsilon_k a_k^\dagger a_k}}$$

$$\langle a_k^\dagger a_k \rangle = \frac{\operatorname{tr} a_k^\dagger a_k \, \mathrm{e}^{-\beta a_k^\dagger a_k \varepsilon_k}}{\operatorname{tr} \mathrm{e}^{-\beta a_k^\dagger a_k \varepsilon_k}} = \left[\mathrm{e}^{\frac{1}{\beta \varepsilon_k}} - \xi \right]$$

$$n_k = \frac{1}{\mathrm{e}^{\beta \varepsilon_k - 1}} \quad \text{bosons}$$

$$n_k = \frac{1}{\mathrm{e}^{\beta \varepsilon_k + 1}} \quad \text{fermions}$$

$$\langle a_k^\dagger a_{k'} \rangle = \frac{\operatorname{tr}\Big(\mathrm{e}^{-\beta H} \, a_k^\dagger \Big)}{Z_k} \, \frac{\operatorname{tr}\big(\mathrm{e}^{-\beta H} a_{k'} \big)}{Z_{k'}}$$

The first factor gives us terms like tr $\langle n_\alpha | a_k^\dagger | n_\alpha \rangle$. Zero.

$$\langle f_1, \ldots, f_k \rangle = \langle f_1^\dagger f_1 \rangle \, \langle f_2^\dagger f_2 \rangle \cdots \langle f_{k'}^\dagger f_{k'} \rangle$$

No other combination gives nonzero.