2008–11–19

Today we will make further use of the symplectic transformations we had for bosons, which we introduced yesterday.

Weakly interacting Bose gas

$$H = \sum_p \frac{p^2}{2\,m} b_p^\dagger b_p + \frac{1}{2\Omega} \sum_{pqk} U_k\, b_{p+k}^\dagger\, b_{q-k}^\dagger\, b_q\, b_p$$

$$N = \sum_p b_p^\dagger b_p$$

$$|\psi_G\rangle = \sum_{\{N_0, n_1, n_2, \ldots\}} c_{N_0, n_1, n_2, \ldots} |N_0, n_1, \ldots, n_k\rangle$$

Lots of particles in the non-interacting ground state, and some particles in other states. The non-interacting ground state $|N_0\rangle$ is not an eigenstate of the interacting Hamiltonian, since the particles can change momentum upon interaction.

The number of particles N, or N_0 for that matter, is not necessarily a good quantum number.

$$b_0^\dagger\, b_0\, |\psi\rangle \approx N_0 |\psi\rangle$$

$$\underbrace{b_0\, b_0^\dagger}_{\approx N_0 - 1} - \underbrace{b_0^\dagger b_0}_{\approx N_0} = 1$$

But $N_0 \sim 10^{23}$, so $N_0 - 1 \approx N_0$:

$$b_0^\dagger b_0 |\psi\rangle \approx b_0 b_0^\dagger |\psi\rangle$$

So $b_0^\dagger$ and b_0 basically commute. The one in the commutator is not significant.

$$b_0^\dagger\, b_0 = N_0 = b_0\, b_0^\dagger \quad \Rightarrow b_0 = \sqrt{N_0}$$

The difference between N_0 and N is proportional to volume, so it is a large number in the absolute. $N - N_0 \propto \Omega\, U_0$.

$$\sum_{kpq} U_k\, b_{p+k}^\dagger b_{q-k}^\dagger\, b_q b_p$$

When momenta match up pairwise:

$$p + k = p \Longleftrightarrow k = 0$$

$$U_0\, b_p^\dagger b_q^\dagger b_q\, b_p$$

"Hartree" term

"Exchange term" or "Fock term"

$$p + k = q, \quad k = q - p$$

$$U_{q-p}\, b_q^\dagger\, b_p^\dagger b_q b_p$$

"Anomalous term"

$$p + k = -q + k \quad p = -q$$

$$U_k b_{-q+k}^\dagger b_{q-k}^\dagger\, b_q\, b_{-q}$$

Let $k = q - p$:

$$U(q-p)\, b_p^\dagger b_{-p}^\dagger b_q b_{-q}$$

$$H_{\text{int}} = \sum_{kpk} \{\}_{\text{no two momenta match}} + []_{\text{the ones we picked out}}$$

$$H = \sum_p \varepsilon_p\, b_p^\dagger b_p + \ldots$$

Hartree:

$$+\frac{1}{2}\frac{U_0}{\Omega}\sum_p b_p^\dagger b_p \sum_q b_q^\dagger b_q = \frac{1}{2}\frac{U_0}{\Omega} N^2$$

Fock:

$$\frac{1}{2\Omega}\sum_{pq} U(p-q)\, b_p^\dagger b_q^\dagger b_q\, b_p$$

Anomalous:

$$\frac{1}{2\Omega}\sum_p U(p-q)\, b_p^\dagger b_{-p}^\dagger b_q\, b_{-q}$$

$$+\frac{1}{2\Omega}\sum_{\substack{pqk \\ \text{no two} \\ \text{momenta} \\ \text{match}}} U_k\, b_{p+k}^\dagger b_{q-k}^\dagger b_q\, b_p$$

Approximation:

Fock:

$$\sim \frac{1}{2\Omega}\sum_p U(p)\, b_p^\dagger b_p\, N_0 \times 2$$

Anomalous

$\frac{1}{2\Omega}\sum_p (U(p)\, b_p^\dagger b_{-p}^\dagger N_0 + b_p\, b_{-p}\, N_0) +$ all terms that are $\mathcal{O}(U_k)$ accounted for, all higher order terms are dropped. (Need two momenta to vanish.)

$$H \approx \frac{U_0}{2\Omega} N^2 + \sum_p \varepsilon_p\, b_p^\dagger b_p + \frac{N_0}{2\Omega}\sum_p U_p \left(2 b_p^\dagger b_p + b_p^\dagger b_{-p}^\dagger + b_p\, b_{-p}\right) =$$

$$= \text{“Effective Hamiltonian”}$$

$$= \frac{1}{2}\sum_{p,\alpha\beta} \psi_{p\alpha}^\dagger(p)\, H_{\alpha\beta}(p)\, \psi_\beta(p) =$$

$$\left[\psi_p = (b_p, b_{-p}^\dagger), \qquad \mu_p = \frac{N_0}{\Omega} U_p = n_0\, U_p\right]$$

$$= \frac{1}{2}\sum_{p\alpha\beta} b_p^\dagger b_{-p} \begin{pmatrix} \varepsilon_p + \mu_p & \mu_p \\ \mu_p & \varepsilon_p + \mu_p \end{pmatrix} \begin{pmatrix} b_p \\ b_{-p}^\dagger \end{pmatrix}$$

Now we do a canonical transformation to construct new operators with energies on the diagonal.

$$\varepsilon = \Gamma \times \text{eigenvalues of } \Gamma H$$

What we are looking for are the eigenvalues of

$$\Gamma H = \begin{pmatrix} \varepsilon_p + \mu_p & \mu_p \\ -\mu_p & -\varepsilon_p - \mu_p \end{pmatrix}$$

$$((\varepsilon_p + \mu_p) - E)(-(\varepsilon_p + \mu_p) - E) + \mu_p^2 = 0$$

$$E^2 - (\varepsilon_p + \mu_p)^2 = \mu_p^2$$

$$E_p^2 = \varepsilon_p^2 + 2\mu_p \varepsilon_p$$

$$E_p = \sqrt{\varepsilon_p^2 + 2\mu_p \varepsilon_p}$$

$$\mu_p = \mu_0 + (\mu_p - \mu_0)$$

$$E_p = \sqrt{\varepsilon_p^2 + 2\mu_0 \varepsilon_p + 2(\mu_p - \mu_0)\varepsilon_p} = \sqrt{\varepsilon_p}\,\sqrt{2\mu_0}\,\sqrt{1 + \frac{2\,(\mu_p - \mu_0)}{2\mu_0} + \frac{|\varepsilon_p|}{2\mu_0}}$$

$$\varepsilon_p = |p|\sqrt{\frac{\mu_0}{m}}\sqrt{1 + \frac{\mu_p - \mu_0}{\mu_0} + \frac{|\varepsilon_p|}{2\mu_0}}$$

Let $c = \sqrt{\frac{\mu_0}{m}}$:

$$\varepsilon_p = |p| c \sqrt{1 + \frac{\mu_p - \mu_0}{\mu_0} + \frac{|\varepsilon_p|}{2\mu_0}}$$

⟨fig.⟩

Feynman.

⟨fig⟩

	initially	
parameter	$\boldsymbol{P}_0 =$	$\boldsymbol{P} + p$ (bug+superfluid excitation)
energy	$\boldsymbol{P}_0^2/2M =$	$\frac{\boldsymbol{P}^2}{2M} + \varepsilon_p$

$$\frac{(\boldsymbol{P} + p)^2}{2M} = \frac{\boldsymbol{P}^2}{2M} + \varepsilon(p)$$

$$\frac{\boldsymbol{P}^2}{2M} + \frac{2\,p \cdot \boldsymbol{P}}{2\,M} + \frac{p^2}{2M} = \frac{\boldsymbol{P}^2}{2M} + \varepsilon(p)$$

$$\frac{p \cdot \boldsymbol{P}}{M} = \varepsilon(p)$$

$$p \cdot V = \varepsilon(p)$$

$$V = \frac{\varepsilon(p)}{p}$$

$\Rightarrow$ superfluid.