# 2008–11–18

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Pathria – Quantum Statistical Mechanics

$$H = \sum_p \varepsilon_p a_p^\dagger a_p + \frac{1}{2} \sum_{x\,y} :V(x-y)\, a_x^\dagger a_x\, a_y^\dagger a_y:$$

The colons mean ordering.

$$H = \sum_p \varepsilon_p a_p^\dagger a_p + \frac{1}{2} \sum_{x\,y} V(x-y)\, a_x^\dagger a_y^\dagger\, a_y a_x$$

$$a_x^\dagger = \sum_p \langle x|p\rangle\; a_p^\dagger = \frac{1}{\sqrt{2\pi\Omega}} \int \mathrm{e}^{\mathrm{i}p\cdot x}\, a_p^\dagger$$

Interaction

$$\frac{1}{(2\pi\Omega)^2} \cdot \frac{1}{2} \sum_{x\,y} \sum_{p_1,\ldots,p_4} \mathrm{e}^{\mathrm{i}p_1\cdot x}\, \mathrm{e}^{\mathrm{i}p_2\cdot y}\, \mathrm{e}^{-\mathrm{i}p_3\cdot y'} \mathrm{e}^{-\mathrm{i}p_4\cdot x}\, V(x-y) \left\{ a_{p_1}^\dagger a_{p_2}^\dagger a_{p_3} a_{p_4} \right\}$$

$$\frac{1}{(2\pi\Omega)^2}\, \frac{1}{2} \sum_{p,\ldots} \sum_{\substack{(x-y)\\ y}} \mathrm{e}^{\mathrm{i}(p_1-p_4)\cdot(x-y)+\mathrm{i}(p_1-p_4)\cdot y+\mathrm{i}(p_2-p_3)\cdot y}\, V(x-y) \left\{ a_{p_1}^\dagger a_{p_2}^\dagger a_{p_3} a_{p_4} \right\}$$

The parts depending on $(x-y)$ and $y$ can be integrated independently.

$$\frac{1}{2} \cdot \frac{1}{2\pi} \cdot \frac{1}{\Omega^2} \sum_p V_{p_1-p_4} \sum_y \mathrm{e}^{\mathrm{i}(p_1+p_2-p_3-p_4)\cdot y} a_{p_1}^\dagger\, a_{p_2}^\dagger\, a_{p_3}\, a_{p_4}$$

where

$$V_p = \frac{1}{2\pi} \sum_x V(x)\, \mathrm{e}^{\mathrm{i}p\cdot x}$$

Interaction:

$$\frac{1}{2} \cancel{\sum_{p_1+p_2-p_3-p_4=0}} V_{p_1-p_4}\, a_{p_1}^\dagger\, a_{p_2}^\dagger a_{p_3}\, a_{p_4}$$

$$\frac{1}{2\Omega} \sum_{k,p,q} V_k\, a_{p+k}^\dagger a_{q-k}^\dagger\, a_q\, a_p$$

$$H = \sum_p \frac{p^2}{2\,m} + \frac{1}{2\Omega} \sum V_k a_{p+k}^\dagger\, a_{q-k}^\dagger\, a_q\, a_p$$

That is the Hamiltonian we will deal with during the next three or four lectures.

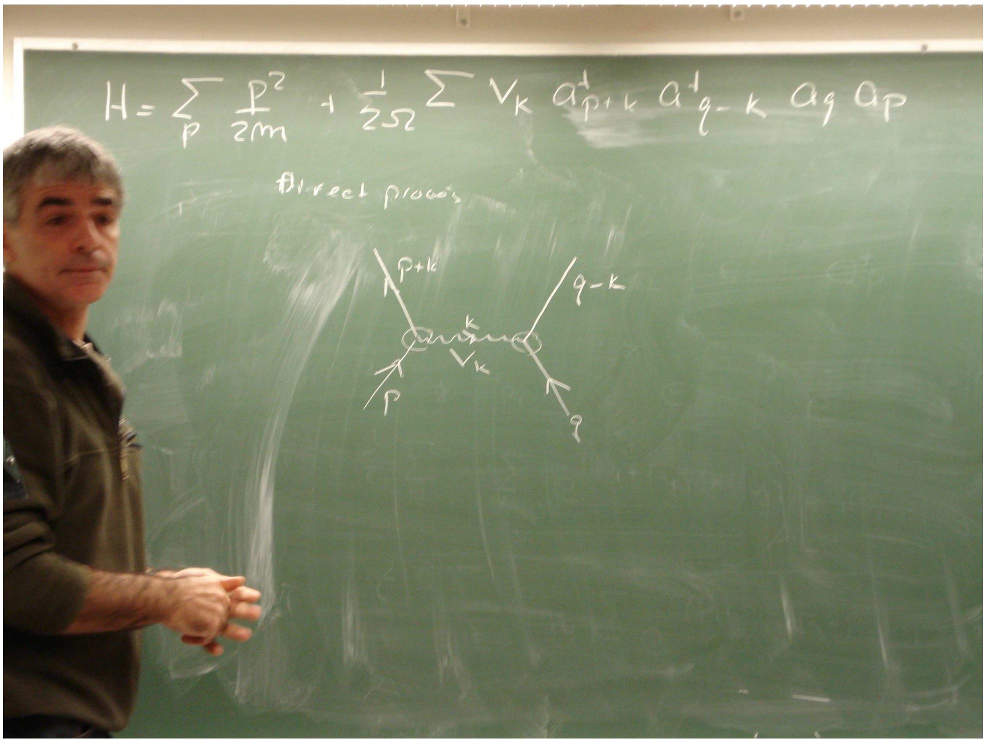


**Figure 1.** Direct process.

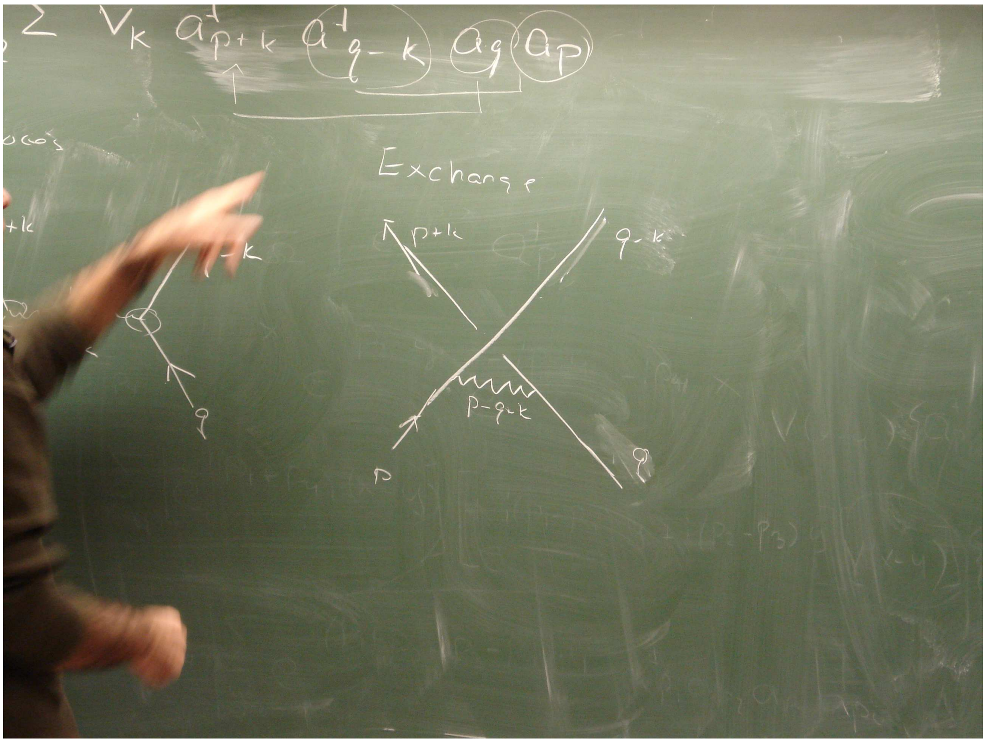


**Figure 2.** Exchange.

$$a_x^\dagger = \sum_p \langle x | p \rangle \, a_p^\dagger$$

Unitary single particle transformation $\Rightarrow$ canonical transformation for fermions and bosons.

Canonical transformations that mix creation and annihilation operators for fermions.

$$\hat{c}_a = U_{aa}\, c_a + U_{ab}\, c_b^{\dagger}$$

$$\hat{c}_b = U_{ba}\, c_a + U_{bb} c_b^{\dagger}$$

Bogoliubov-Valatin transformation.

$$\psi = (c_a, c_b^{\dagger}), \quad \psi^{\dagger} = (c_a^{\dagger}, c_b)$$

$$\alpha = (a, b), \quad \alpha' = (a', b')$$

$$\{\psi_{\alpha}, \psi_{\alpha'}\} = \delta_{\alpha\alpha'}$$

$$\hat{\psi}_{\alpha} = \sum_{\alpha'} U_{\alpha\alpha'} \psi_{\alpha'}, \quad U = \begin{pmatrix} U_{aa} & U_{ab} \\ U_{ba} & U_{\mathrm{bb}} \end{pmatrix}$$

$$\delta_{\alpha\beta} = \hat{\psi}_{\alpha}\, \hat{\psi}_{\beta}^{\dagger} + \hat{\psi}_{\beta}^{\dagger} \hat{\psi}_{\alpha} = \sum_{\alpha'} U_{\alpha\alpha'} U_{\beta\beta'}^{*} \underbrace{\left( \psi_{\alpha'} \psi_{\beta'}^{\dagger} + \psi_{\beta'}^{\dagger} \psi_{\alpha'} \right)}_{\delta_{\alpha'\beta'}} = \sum U_{\alpha\alpha'} U_{\alpha'\beta}^{\dagger}$$

$$\mathbf{1} = U U^{\dagger}$$

$$U = \begin{pmatrix} U_{aa} & U_{ab} \\ U_{ab}^{\dagger} & U_{bb} \end{pmatrix}$$

$U$ unitary $\Rightarrow$ canonical fermion transformation. [There must be a mistake here. The $U$ above is not, in general, unitary. There should be $-U_{ab}^{\dagger}$ instead of $U_{ab}^{\dagger}$ on the last line, I think.]

Canonical transformation for bosons.

$$\hat{b}_a = U_{aa}\, b_a + U_{ab}\, b_b^{\dagger}$$

$$\hat{b}_b = U_{ba}\, b_a + U_{bb} b_b^{\dagger}$$

Collect into $2(k)$ component object

$$\phi = (b_a, b_b^{\dagger}), \quad \phi^{\dagger} = (b_a^{\dagger}, b_b)$$

$$\left[ \phi_{\alpha}, \phi_{\beta}^{\dagger} \right] = \left( \begin{array}{c|c} \mathbf{1} & 0 \\ \hline 0 & -\mathbf{1} \end{array} \right) = \Gamma = \left( \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots \end{array} \right)$$

$$\left[ \hat{\phi}_{\alpha}, \hat{\phi}_{\beta}^{\dagger} \right] = \sum_{\alpha', \beta'} U_{\alpha\alpha'} U_{\beta\beta'}^{*} \left[ \phi_{\alpha'}, \phi_{\beta'}^{\dagger} \right]$$

$$\Gamma_{\alpha\beta} = \sum_{\alpha'\beta'} U_{\alpha\alpha'} \Gamma_{\alpha'\beta'} U_{\beta'\beta}^{\dagger}$$

$\Gamma = U\, \Gamma\, U^{\dagger} \Longleftrightarrow$ canonical transformation.

$$\Gamma = U\, \Gamma\, U^{\dagger}$$

Multiply by $\Gamma$:

$$\mathbf{1} = U\left(\Gamma\, U^{\dagger}\, \Gamma\right) \quad \Rightarrow \quad U^{-1} = \Gamma\, U^{\dagger}\, \Gamma$$

Possibility: If $b^{\dagger}, b$ are not mixed:

$$\begin{pmatrix} U & \\ & U' \end{pmatrix} \text{ where } U \text{ and } U' \text{ are unitary.}$$

This implies new properties for the eigenvalue spectrum.

$$H = \sum_{\alpha,\beta} \phi_{\alpha}^{\dagger} H_{\alpha\beta}\, \phi_{\beta}$$

$$H_{\alpha\beta} = \begin{pmatrix} H_{aa} & H_{ab} \\ H_{ba} & H_{bb} \end{pmatrix}$$

Diagonalize $H$

$$\mathcal{E} = U^{\dagger} H\, U$$

$$\Gamma\mathcal{E} = \left(\Gamma U^{\dagger}\, \Gamma\,\right)\Gamma\, H\, U = U^{-1}\, \Gamma H\, U$$

$$\Gamma H = \begin{pmatrix} H_{aa} & H_{ab} \\ -H_{ba} & -H_{bb} \end{pmatrix}$$

Prescription is: diagonalize $\Gamma H$.

$$\begin{pmatrix} \varepsilon_1 & 0 & 0 & 0 \\ 0 & \varepsilon_2 & 0 & 0 \\ 0 & 0 & -\varepsilon_3 & 0 \\ 0 & 0 & 0 & -\varepsilon_4 \end{pmatrix}$$

**Theory of weakly interacting Bose gas.** Pathria

$V_k$ small, bosons.

Non-interacting limit.

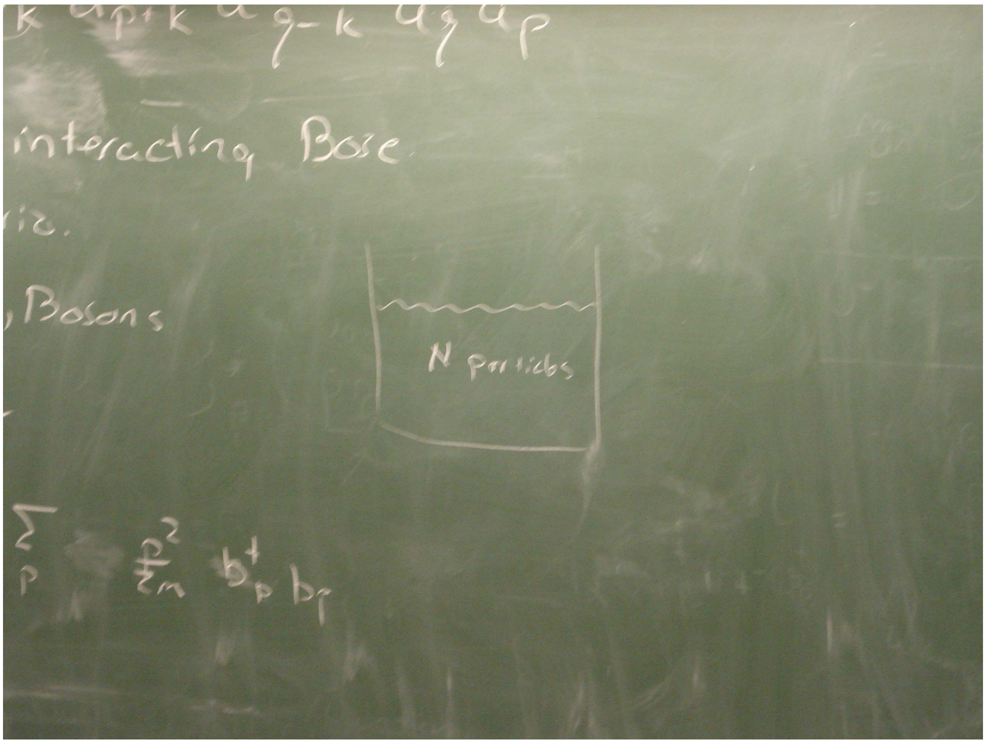


**Figure 3.**

If I lower the temperature, what is going to be the ground state?

$$H = \sum_p \frac{p^2}{2m} b_p^\dagger b_p$$

The ground state

$$|G\rangle = |n_0, n_1, \ldots, n_k\rangle = |N\rangle$$

with $N \sim 10^{23}$.

At finite temperature, mixture of

$$|N\rangle, \quad |N-1, 1_k\rangle, \quad |N-2, 1_k, 1_{k'}\rangle$$

The ground state is not well described as a state with a fixed number of particles. Better:

$$|\psi_0\rangle = \sum_{N'} \mathrm{e}^{\mathrm{i}\varphi_{N'}} |N'\rangle\, \mathrm{e}^{-(N-N')^2/\gamma}$$

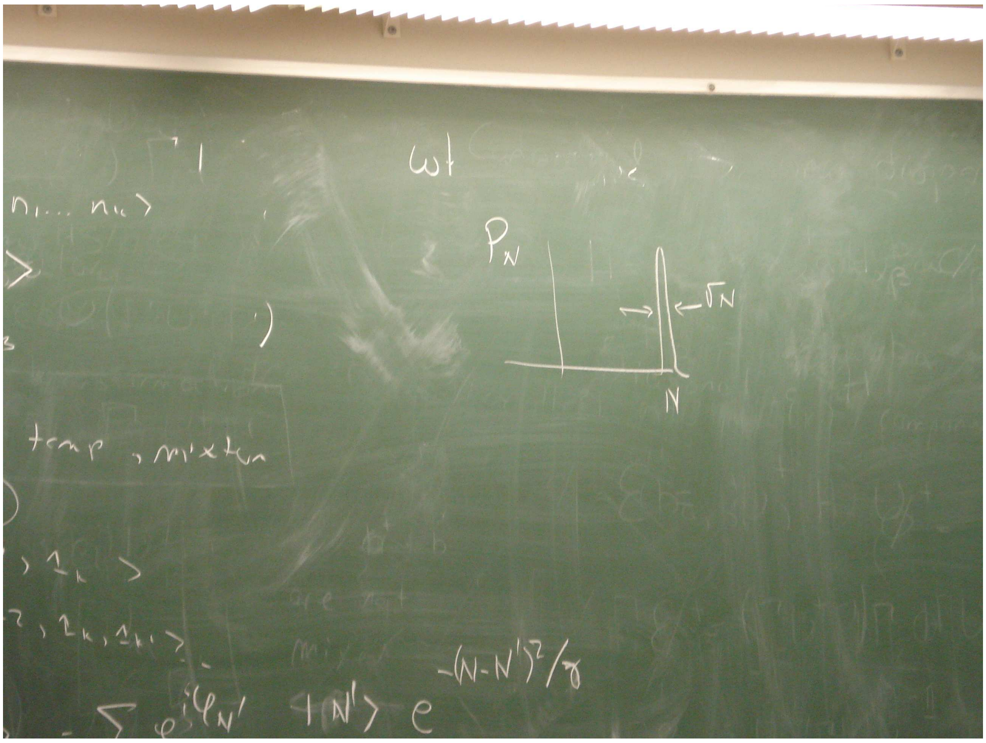


**Figure 4.**

$$|G\rangle \neq |N_0\rangle$$

$$|G\rangle = \sum_{\{n_k, N_0\}} |N_0, n_1, n_2, \ldots\rangle C_{N_0, n_1, \ldots, n_k}$$