2008–11–14

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \ldots$$

$$a_\lambda^\dagger |n_\lambda, n_1, \ldots, n_k\rangle = \sqrt{|n_\lambda + \xi|}\, |(n_\lambda + 1), n_2, \ldots\rangle$$

$$a_\lambda |n_\lambda, n_1, \ldots, n_k\rangle = \sqrt{n_\lambda}\, |(n_\lambda - 1), n_2, \ldots\rangle$$

For bosons:

$$[b_\lambda, b_{\lambda'}^\dagger] = \delta_{\lambda\lambda'}$$

For fermions:

$$c_\lambda^\dagger |0\rangle = |1_\lambda\rangle, \quad c_\lambda^\dagger |1_\lambda\rangle = 0$$

$$c_\lambda |0\rangle = 0, \quad c_\lambda |1_\lambda\rangle = |0\rangle$$

$$c_\lambda\, c_\lambda^\dagger\, |0\rangle + c_\lambda^\dagger c_\lambda |0\rangle = |0\rangle$$

$$c_\lambda c_\lambda^\dagger |1_\lambda\rangle + c_\lambda^\dagger c_\lambda |1_\lambda\rangle = |1_\lambda\rangle$$

$$c_\lambda\, c_\lambda^\dagger + c_\lambda^\dagger c_\lambda = \mathbf{1}$$

$$\{c_\lambda, c_{\lambda'}^\dagger\} = \mathbf{1}\, \delta_{\lambda\lambda'}$$

$$\left[a_\lambda, a_{\lambda'}^\dagger\right]_\xi = \delta_{\lambda\lambda'}$$

Bosons:

$$|n_\lambda\rangle$$

$$\begin{pmatrix} |0\rangle \\ |1\rangle \\ |2\rangle \\ \vdots \end{pmatrix}$$

$$b_\lambda^\dagger |n_\lambda\rangle \;\text{ — what matrix is that?}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ \vdots & 0 & \sqrt{3} \\ 0 & \vdots & \vdots \end{pmatrix} \begin{pmatrix} |0\rangle \\ |1\rangle \\ |2\rangle \\ \vdots \end{pmatrix}$$

Fermion:

$$c_\lambda^\dagger = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad c_\lambda = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Unitary transformation.

Fermions + Bosons

$$a_\lambda = \sum_{\lambda'} U_{\lambda\lambda'} \hat{a}_{\lambda'}$$

$$\begin{pmatrix} c_\uparrow \\ c_\downarrow \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} c'_\downarrow \\ c'_\uparrow \end{pmatrix}$$

$$c_\lambda^\dagger = U_{11}^* {c'_\downarrow}^\dagger + U_{12}^* {c'_\lambda}^\dagger$$

$$a_\lambda^\dagger = \sum_\lambda U_{\lambda\lambda'}^* \hat{a}_{\lambda'}$$

Check:

$$a_\alpha a_\beta^\dagger \pm a_\beta^\dagger a_\alpha \overset{?}{=\!=} \delta_{\alpha\beta}$$

$$U_{\alpha\lambda} U_{\beta\lambda'}^\dagger \underbrace{\left(\hat{a}_\lambda \hat{\alpha}_{\lambda'}^\dagger \pm \hat{a}_{\lambda'} \hat{\alpha}_\lambda \right)}_{=\delta_{\lambda\lambda'}} \overset{?}{=\!=} \delta_{\alpha\beta}$$

$$\sum_{\lambda\lambda'} U_{\alpha\lambda} U_{\lambda'\beta}^\dagger \delta_{\lambda\lambda'} = \delta_{\alpha\beta} \quad (U U^\dagger = \mathbf{1})$$

$$\Rightarrow \text{ canonical transformation}$$

$$a_{\lambda'} a_\lambda^\dagger - \xi a_\lambda^\dagger a_{\lambda'} = \delta_{\lambda\lambda'}$$

What if ξ was a complex number? When we take the Hermitian adjoint there will be a problem.

We need some algebra. Bosons:

$$[b_\lambda^\dagger, b_{\lambda'}^\dagger b_{\lambda'}] = b_{\lambda'}^\dagger \left[b_\lambda^\dagger, b_{\lambda'} \right] = -\, b_\lambda^\dagger \delta_{\lambda\lambda'}$$

Fermions:

$$\left[c_\lambda^\dagger, c_{\lambda'}^\dagger c_{\lambda'} \right] = c_\lambda^\dagger c_{\lambda'}^\dagger c_{\lambda'} - c_{\lambda'}^\dagger c_{\lambda'} c_\lambda^\dagger =$$

$$= -\, c_{\lambda'}^\dagger c_\lambda^\dagger c_{\lambda'} - c_{\lambda'}^\dagger c_{\lambda'} c_\lambda^\dagger = -\, c_{\lambda'}^\dagger \left(\delta_{\lambda\lambda'} - \cancel{c_\lambda} c_\lambda^\dagger \right) - c_{\lambda'}^\dagger \cancel{c_{\lambda'}} c_\lambda^\dagger = -\, c_{\lambda'}^\dagger$$

Thus, for both bosons and fermions,

$$[a_\lambda^\dagger, n_{\lambda'}] = -\, a_\lambda^\dagger \delta_{\lambda\lambda'}$$

$H |\lambda\rangle = \varepsilon_\lambda |\lambda\rangle$ (single particle Hamiltonian):

$$H |n_1, n_2, \ldots, n_k\rangle = (n_1 \varepsilon_1 + n_2 \varepsilon_2 + \cdots + n_k \varepsilon_k) |n_1, \ldots, n_k\rangle$$

$$H = \sum_k n_k \varepsilon_k = \sum_k a_k^\dagger a_k \varepsilon_k = H$$

Second quantisation.

Bosons

$$\left[b_\lambda^\dagger, b_{\lambda'}^\dagger b_{\lambda'}\right] = b_{\lambda'}^\dagger \left[b_\lambda^\dagger, b_\lambda\right] = -\, b_\lambda^\dagger \delta_{\lambda\lambda'}$$

$$\mathrm{i}\hbar \frac{\mathrm{d}a_\lambda^\dagger}{\mathrm{d}t} = \left[a_\lambda^\dagger, H\right] = \sum_{\lambda'} \varepsilon_{\lambda'} \left[a_\lambda^\dagger, a_{\lambda'}^\dagger a_{\lambda'}\right] = \sum_{\lambda'} \varepsilon_{\lambda'} \left(-\, a_{\lambda'}^\dagger \right) \delta_{\lambda\lambda'} = -\, \varepsilon_\lambda\, a_\lambda^\dagger$$

$$a_\lambda^\dagger(t) = \mathrm{e}^{\mathrm{i}\varepsilon_\lambda t/\hbar}\, a_\lambda^\dagger$$

$$H = \sum_n \varepsilon_n c_n^\dagger c_n$$

$$c_n = \sum_{nx} U_{nx}\, c_x = \sum_x \langle \lambda | x \rangle\, c_x$$

$$U_{\lambda x} \equiv \langle \lambda | x \rangle$$

$$H^N = \sum_{\lambda x y} \varepsilon_\lambda \langle \lambda | x \rangle^* c_x^\dagger \langle \lambda | y \rangle c_y = \sum_{xy} c_x^\dagger c_y \sum_\lambda \langle x | \lambda \rangle \varepsilon_\lambda \langle \lambda | y \rangle = \sum_{xy} \langle x | H^1 | y \rangle$$

In real space representation:

$$H = \sum_{xy} c_x^\dagger \langle x | H | y \rangle c_y$$

$$H = \sum_{xy} c_x^\dagger \frac{\hbar^2}{2\,m} \frac{\partial^2}{\partial x^2} c_x + \sum_x V_1(x)\, c_x^\dagger c_x$$

$$= \sum_p c_p^\dagger\, c_p \frac{p^2}{2m} + \sum_{pp'} \tilde{V}_1(p - p')\, c_p^\dagger\, c_{p'}$$

$$\sum_x V(x)\, c_x^\dagger\, c_x = \sum_{xpp'} V(x) c_p^\dagger c_{p'}\, \langle p | x \rangle \langle x | p' \rangle =$$

$$= \frac{1}{2\pi\hbar} \sum_{xpp'} V(x) c_p^\dagger c_{p'} \cdot \mathrm{e}^{\mathrm{i}p\cdot x/\hbar} \cdot \mathrm{e}^{-\mathrm{i}p'\cdot x/\hbar} = \sum_{pp'} c_p^\dagger c_{p'} \left(\frac{1}{2\pi\hbar} \sum_x \mathrm{e}^{\mathrm{i}x(p-p')/\hbar}\, V(x) \right)$$

$$H\, c_p^\dagger = \frac{p^2}{2m}\, c_p^\dagger + \sum_{p'} \tilde{V}(p - p')\, c_{p'}^\dagger$$

Interactions

$$V = \frac{1}{2} \sum_{xy} V(x - y) \underbrace{(n_x\, n_y - n_x \delta_{xy})}_{\text{number of pairs that interact}}$$

$$= \frac{1}{2} \sum_{xy} V(x - y) c_x^\dagger c_x c_y^\dagger c - \delta_{xy}\, c_x^\dagger c_x = \frac{1}{2} \sum_{xy} V(x - y) \left(\xi c_x^\dagger c_y^\dagger\, c_x\, c_y + c_x^\dagger \delta_{xy}\, c_y - \delta_{xy} c_x^\dagger c_x \right) =$$

$$= \frac{1}{2} \sum_{xy} V(x - y)\, c_x^\dagger c_y^\dagger c_y c_x$$

This is normal ordered.

$$|0\rangle = \text{zero particles}$$

$$|1\rangle = \sum_x \psi(x)\, c_x^\dagger |0\rangle$$

$$|2\rangle = \sum_x \psi(x,y)\, c_x^\dagger c_y^\dagger |0\rangle$$

$$V|0\rangle = 0$$

$$|3\rangle = \sum_{xyz} \psi(x,y,z) c_x^\dagger c_y^\dagger c_z^\dagger |0\rangle$$

Interacting Hamiltonian

$$H = \underbrace{\sum_{\alpha\beta} \langle\alpha|H|\beta\rangle c_\alpha^\dagger c_\beta}_{\text{free part}} + \underbrace{\frac{1}{2}\sum_{\alpha\beta\delta\gamma} \langle\alpha\beta|V|\gamma\delta\rangle c_\alpha^\dagger c_\beta^\dagger c_\gamma c_\delta}_{\text{interactions}}$$

Fourier transform

$$V = \frac{1}{2}\sum_{xy} V(x-y)\, c_x^\dagger c_y^\dagger c_y c_x$$

$$V = \frac{1}{2}\sum_{pqk} V(k) c_{p+k}^\dagger c_{q-k}^\dagger c_q c_p$$