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The intent of this course is that you get an overview of more advanced methods in quantum mechanics. It is a bit of an overview, we can't get as deep into stuff as we would like.

At this point Sakurai will not be of much use. What we are going to discuss today and the next few lectures, is multiparticle physics and "second quantisation".

There are two fundamental types of particle: Bosons – interactions $\Rightarrow$ superfluid. Fermions – interactions $\Rightarrow$ superconductor. This is macroscopic phenomena that must be understood using quantum mechanics. We will try to treat these without introducing too much mathematical formalism.

Multiparticle systems

We have N particles, where each particle can be in one of several states. We label the states by α: $|\alpha_1, \ldots, \alpha_N) = |\alpha_1\rangle \otimes |\alpha_2\rangle \otimes \cdots \otimes |\alpha_N\rangle$. We have put a parenthesis on the right: $|\alpha_1, \ldots, \alpha_n)$, instead of an angle bracket as in $|\alpha_1\rangle$, because of some normalisation problems.

$$\psi_{\alpha_1, \ldots, \alpha_N}(r_1, \ldots, r_N) = (r_1, \ldots, r_N | \alpha_1, \ldots, \alpha_N)$$

Normalisation:

$$\sum_{\alpha_1, \ldots, \alpha_N} |\alpha_1, \ldots, \alpha_N)(\alpha_1, \ldots, \alpha_N| = 1$$

Identical particles, which are fermions:

$$|\alpha_1, \alpha_2, \ldots, \alpha_N) = (-1)\, |\alpha_2, \alpha_1, \alpha_3, \ldots, \alpha_N)$$

Identical particles, which are bosons:

$$|\alpha_1, \alpha_2, \ldots, \alpha_N) = (+1)\, |\alpha_2, \alpha_1, \alpha_3, \ldots, \alpha_N)$$

$$(-1)^P = \begin{cases} -1 & \text{if } P \text{ is an odd permutation} \\ +1 & \text{if } P \text{ is even} \end{cases}$$

$$\xi^P\text{:} \quad \xi = 1 \text{ for bosons,} \quad = -1 \text{ for fermions}$$

$$|\alpha_1, \ldots, \alpha_n) = (\xi)^P\, |\alpha_{p_1}, \alpha_{p_2}, \ldots, \alpha_{p_N})$$

A permutation (1, 2, 3, 4) $\rightarrow$ (4, 1, 2, 3) — How many simple exchanges are needed to get from one to the other? It is always either odd or even, so that's what we mean by an odd or even permutation.

Definition:

$$|\alpha_1, \ldots, \alpha_N\rangle = \frac{1}{\sqrt{N!\, \Pi(n_\alpha!)}} \sum_P \xi^P\, |\alpha_{p_1}, \ldots, \alpha_{p_N})$$

Occupation number of the states

$$\sum_\alpha n_\alpha = N$$

$n_\alpha =$ degeneracy of all states α.

Fermion: example: 2 fermions in 2 different states.

$$|\alpha_1,\alpha_2\rangle = \frac{1}{\sqrt{2}}\left(\ |\alpha_1,\alpha_2) - |\alpha_2,\alpha_1)\ \right)$$

Let us check the normalisation:

$$\langle\alpha_1,\alpha_2|\alpha_1,\alpha_2\rangle = \frac{1}{2}\left(\ (\alpha_1,\alpha_2| - (\alpha_2,\alpha_1|\ \right)\left(\ |\alpha_1,\alpha_2) - |\alpha_2,\alpha_1)\ \right) = 1$$

Why do we do this in the first place? We want to make sure that the state $|\alpha_1,\ldots,\alpha_N\rangle$ is inherently antisymmetric. For fermions $n_\alpha = 1$: there can only be one fermion in each state.

$N=3$, bosons. We are going to look at $|\alpha_1,\alpha_1,\alpha_2\rangle$

$$|\alpha_1,\alpha_1,\alpha_2\rangle = \frac{2!}{\sqrt{3!\times 2!}}\left(\ |\alpha_1,\alpha_1,\alpha_2) + |\alpha_1,\alpha_2,\alpha_1) + |\alpha_2,\alpha_1,\alpha_1)\ \right)$$

The 2! in the numerator comes from the fact that we only write down three terms: the original formula would have us write down six terms, but interchanging α_1 and α_1 we get the same thing again.

$$|\alpha_1,\alpha_1,\alpha_2\rangle = \frac{1}{\sqrt{3}}\left(\ |\alpha_1,\alpha_1,\alpha_2) + |\alpha_1,\alpha_2,\alpha_1) + |\alpha_2,\alpha_1,\alpha_1)\ \right)$$

This is properly normalised.

Distinguishable particles. Say that we have the states α_1, α_1, α_2 that the distinguishable particles can occupy. Specifying that two particles go into α_1 and one goes into α_2 completely specifies the state.

We have red, blue and green:

$$\begin{array}{ll} \texttt{r}\ 1,2 & \\ \texttt{b}\ 1,2 & 2^3 \\ \texttt{g}\ 1,2 & \end{array}$$

if we don't specify how many particles are in each state.

Fermions:

$$\psi_{\alpha_1,\ldots,\alpha_N}(r_1,\ldots,r_N) = \frac{1}{\sqrt{N!}}\sum\ \xi^P\langle r_1|\alpha_{p_1}\rangle\langle r_2|\alpha_{p_2}\rangle\cdots\langle r_N|\alpha_{p_N}\rangle$$

$$\psi_{\alpha_1,\ldots,\alpha_N}(r_1,\ldots,r_N) = \frac{1}{\sqrt{N!}}\det\begin{pmatrix} \psi_{\alpha_1}(r_1) & \psi_{\alpha_1}(r_2) & \cdots & \psi_{\alpha_1}(r_N) \\ \psi_{\alpha_2}(r_1) & \cdots & & \psi_{\alpha_2}(r_N) \\ \vdots & & & \vdots \\ \psi_{\alpha_N}(r_1) & \psi_{a_N}(r_2) & \cdots & \psi_{\alpha_N}(r_N) \end{pmatrix}$$

This is the Slater determinant.

For bosons:

$$= \frac{1}{\sqrt{N!}}\,\mathrm{Perm}$$

The definition of the permanent of A differs from that of the determinant of A in that the signatures of the permutations are not taken into account:

$$\mathrm{Perm}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\,d + b\,c, \quad \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\,d - b\,c$$

$$|\alpha_1,\alpha_1,\alpha_2\rangle = |\alpha_1,\alpha_2,\alpha_1\rangle$$

We are going to the particle number representation: $|2_{\alpha_1}, 1_{\alpha_2}\rangle$. This is exactly equivalent to the above. In general: $|n_{\alpha_1}, n_{\alpha_2}, ..., n_{\alpha_k}\rangle$.

$$|n_{\alpha_1}, n_{\alpha_2}, ..., n_{\alpha_k}\rangle = |\underbrace{a_1, ..., a_1}_{n_{\alpha_1}}, \underbrace{\alpha_2, ..., \alpha_2}_{n_{\alpha_2}}, ...\rangle$$

The particle number representation

$$|n_{\alpha_1}, n_{\alpha_2}, ..., n_{\alpha_k}\rangle \quad \text{Bosons}$$

$$|1_{\alpha_1}, ..., 1_{\alpha_N}\rangle \quad \text{Fermions}$$

$$|n_{\alpha_1}, n_{\alpha_2}, ..., n_{\alpha_k}\rangle = |n_{\alpha_2}, n_{\alpha_1}, n_{\alpha_3}, ..., n_{\alpha_k}\rangle \quad \text{Bosons}$$

$$|1_{\alpha_1}, 1_{\alpha_2}, ...\rangle = -1\, |1_{\alpha_2} 1_{\alpha_1} 1_{\alpha_3}, ...\rangle \quad \text{Fermions}$$

["It is correct by definition"]

Now we are going to introduce the *Fock-space*. It is a Hilbert space that is not confined to a fixed number of particles:

$$\mathcal{F} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_\infty$$

Ex: 1-sate, ε. Fermions. We can have zero particles in the state: $|0\rangle$. $|1_\varepsilon\rangle$. The Fock-space of one state is two dimensional:

$$\begin{pmatrix} 0 \\ 1_\varepsilon \end{pmatrix}$$

If we had two states, what is the Fock-space then?

$$\begin{pmatrix} 0 \\ 1_{\varepsilon_1} \\ 1_{\varepsilon_2} \\ 1_{\varepsilon_1} 1_{\varepsilon_2} \end{pmatrix} = \begin{pmatrix} 0 \\ 1_{\varepsilon_1} \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1_{\varepsilon_2} \end{pmatrix}$$

$$\mathcal{F} = \mathcal{F}_{\varepsilon_1} \otimes \mathcal{F}_{\varepsilon_2}$$

Let us see what the Fock-space looks like for bosons:

1-state:

$\mathcal{H}_0$: $|0\rangle$, $\quad \mathcal{H}_1$: $|1_\varepsilon\rangle$, $\quad \mathcal{H}_2$: $|2_\varepsilon\rangle$, $\quad ...,\quad \mathcal{H}_n$: $|n_\varepsilon\rangle$, $\quad ...$

The Fock-space is infinite dimensional.

2-states:

$\mathcal{H}_0$: $|0\rangle$, $\quad \mathcal{H}_1$: $|1_{\varepsilon_1}\rangle, |1_{\varepsilon_2}\rangle$, $\quad \mathcal{H}_2$: $|2_{\varepsilon_1}\rangle, |2_{\varepsilon_2}\rangle, |1_{\varepsilon_1}, 1_{\varepsilon_2}\rangle$, $\quad ...$

$$= \mathcal{F}_{\varepsilon_1} \otimes \mathcal{F}_{\varepsilon_2}$$

Creation and annihilation operators

$$\text{“}\quad a_\lambda^\dagger |n_{\alpha_1}, ..., n_\lambda, ..., n_{\alpha_k}\rangle \propto |n_\alpha, ..., (n_\lambda + 1), ..., n_{\alpha_k}\rangle \quad\text{”}$$

$$a_{\alpha_2}^\dagger |n_{\alpha_1}, n_{\alpha_2}, n_{\alpha_3}\rangle \propto |n_{\alpha_1}, (n_{\alpha_2} + 1), n_{\alpha_3}\rangle$$

We are going to manipulate only the leftmost element. Other elements can be obtained from there by permutations. More precisely:

$$a_\lambda^\dagger|n_\lambda, n_2, ..., n_k\rangle = \sqrt{|n_\lambda + \xi|}\, |n_{\lambda+1}, n_2, ..., n_k\rangle$$

$$\alpha_\lambda|n_\lambda, n_2, ...n_k\rangle = \sqrt{n_\lambda}\, |n_\lambda - 1, n_2, ..., n_k\rangle$$

$$a_\lambda^\dagger|n_\lambda\rangle = \sqrt{n_\lambda + 1}\, |n_\lambda + 1\rangle$$

$$a_\lambda|n_\lambda\rangle = \sqrt{n_\lambda}\, |n_\lambda - 1\rangle$$

$$[a_\lambda, a_\lambda^\dagger] = 1$$

$$n_\lambda = a_\lambda^\dagger a$$