2008–11–07

On Tuesday there will be an opportunity to ask questions about the homework.

A note on two-body scattering: You have to be careful if the particles are identical. What we have done is $A+B\longrightarrow A+B$ for $A\neq B$.

$$\boldsymbol{r}=\boldsymbol{r}_A-\boldsymbol{r}_B$$

$$\psi(\boldsymbol{r})=\frac{1}{(2\pi)^{3/2}}\left(\mathrm{e}^{\mathrm{i}\boldsymbol{r}\cdot\boldsymbol{p}}+f(\theta,E)\,\frac{\mathrm{e}^{\mathrm{i}pr}}{r}\right)$$

For identical particles we have to do the same thing with A and B interchanged. For spin zero: $\psi(\boldsymbol{r})+\psi(-\boldsymbol{r})$, $\theta\rightarrow\pi-\theta$.

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}=|f(\theta)+f(\pi-\theta)|^2=|f(\theta)|^2+|f(\pi-\theta)|^2+2\,\mathrm{Re}(f(\theta)\,f^*(\pi-\theta))$$

For $\theta=\frac{\pi}{2}$ we get constructive interference. This could be e.g. α particles colliding with the nuclei of a helium gas.

Protons and neutrons are spin $\frac{1}{2}$, so what if $A=B$ with spin $\frac{1}{2}$?

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}=\frac{1}{4}\left\{1\cdot\underbrace{|f(\theta)+f(\pi-\theta)|^2}_{S=0}+3\cdot\underbrace{|f(\theta)-f(\pi-\theta)|^2}_{S=1}\right\}$$

One $S=0$ singlet: $\frac{1}{\sqrt{2}}(|{\uparrow\downarrow}\rangle-|{\downarrow\uparrow}\rangle)$, three $S=1$: $|{\downarrow\downarrow}\rangle,|{\uparrow\uparrow}\rangle,\frac{1}{\sqrt{2}}(|{\uparrow\downarrow}\rangle+|{\downarrow\uparrow}\rangle)$.

$$A+B\rightarrow A+B,\quad A\neq B$$

$$\langle\boldsymbol{k}|T|\boldsymbol{k}'\rangle\overset{\text{parity}}{\longrightarrow}\,?$$

What happens to $|\boldsymbol{k}\rangle$ under parity? Classically $\boldsymbol{k}=m\,\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\rightarrow-\boldsymbol{k}$.

$$\langle\boldsymbol{k}|T|\boldsymbol{k}'\rangle\overset{\text{parity}}{\longrightarrow}\langle-\boldsymbol{k}|T|-\boldsymbol{k}'\rangle$$

$$\underset{\text{reversal}}{\overset{\text{time}}{\longrightarrow}}\langle-\boldsymbol{k}|T|-\boldsymbol{k}'\rangle$$

$$\langle\boldsymbol{k}|T|\boldsymbol{k}'\rangle\overset{\text{PT}}{\longrightarrow}\langle\boldsymbol{k}'|T|\boldsymbol{k}\rangle$$

$$[x,p]=\mathrm{i}$$

$$x\rightarrow\Omega^{-1}\,x\,\Omega=x,\quad p\rightarrow\Omega^{-1}p\,\Omega=-p$$

In parity we change the sign of both, but with time reversal we only change the sign of p.

$$\left[\Omega^{-1}\,x\,\Omega,\,\Omega^{-1}\,p\,\Omega\right]=\Omega^{-1}\,\mathrm{i}\,\Omega$$

$$[x,-p]=-i$$

$$\Omega(a\psi_1+b\,\psi_2)=a^*\,\Omega\,\psi_1+b^*\,\Omega\,\psi_2$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(\boldsymbol{k}\rightarrow\boldsymbol{k}')=\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(\boldsymbol{k}'\rightarrow\boldsymbol{k})$$

This is called the *detailed balance*. (The detailed balance is really a more general concept.)

What we have treated so far is the quantum theory of elastic scattering. If you go to heigher energy, you can get pair creation. This sort of processes is very difficult (not to say impossible) to handle in this formalism. This is necessarily a relativistic process.

$$e^- + A_0 \to e^- + A_n$$

where e^- is an electron and A_0 (A_n) is a generic atom in the ground state (n-th excited state). The energy of the incoming electron and energy of the outgoing electron, are not going to be the same. $k = |\boldsymbol{k}| < k' = |\boldsymbol{k}'|$, where $\boldsymbol{k}$ is the momentum of the incoming electron and $\boldsymbol{k}'$ the momentum of the outgoing electron. With the Born approximation we had:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left|(2\pi)^2 m \langle \boldsymbol{k}'|V|\boldsymbol{k}\rangle\right|^2$$

What do we need to change to make this applicable to this case?

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{k'}{k}\left|(2\pi)^2 m \langle \boldsymbol{k}', n|V|\boldsymbol{k}, 0\rangle\right|^2$$

Remember $\boldsymbol{J}_{\mathrm{in}} \propto k$, $\boldsymbol{J}_{\mathrm{out},r} \propto k'$. How do we compute this?

$$\langle \boldsymbol{k}', n|V|\boldsymbol{k}, 0\rangle$$

The incoming electron will feel a potential from the nucleus, and from all the electrons in the atoms:

$$V = -\frac{Z e^2}{r^2} + \sum_{i=1}^{Z} \frac{e^2}{|\boldsymbol{y}_i - \boldsymbol{x}|^2}$$

$$\langle \boldsymbol{k}', n|V|\boldsymbol{k}, 0\rangle = \int \mathrm{d}^3\boldsymbol{x} \frac{\mathrm{e}^{\mathrm{i}(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{x}}}{(2\pi)^{3/2}} \langle n|V|0\rangle$$

where $\langle n|V|0\rangle$ would be the integral

$$\langle n|V|0\rangle = \int \mathrm{d}\boldsymbol{y}_1 \ldots \mathrm{d}\boldsymbol{y}_z\, \psi_n^*(\boldsymbol{y}_1, \ldots, \boldsymbol{y}_z)\left(-\frac{Z e^2}{r^2} + \sum_{i=1}^{Z} \frac{e^2}{|\boldsymbol{y}_i - \boldsymbol{x}|^2}\right)\psi_0(\boldsymbol{y}_1, \ldots, \boldsymbol{y}_z) =$$

$$= -\frac{Z e^2}{r}\delta_{n0} + \mathrm{mess}(\boldsymbol{x})$$

Setting $\boldsymbol{q} = \boldsymbol{k} - \boldsymbol{k}'$, $q^2 = |\boldsymbol{k} - \boldsymbol{k}'| = k^2 + {k'}^2 - 2\, k\, k' \cos\theta$.

$$\langle \boldsymbol{k}', n|V|\boldsymbol{k}, 0\rangle = \int \mathrm{d}^3\boldsymbol{x} \frac{\mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{x}}}{(2\pi)^{3/2}} \langle n|V|0\rangle = -\frac{4\pi Z e^2}{(2\pi)^{3/2} q^2}\delta_{n0} + \mathrm{mess}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = 4 m^2 \frac{Z^2 e^4}{q^4} \frac{k'}{k} |-\delta_{n,0} + F_n(q)|^2$$

where $F_n(q)$ is called the atomic form factor.

We go back to the old Born approximation, but with a twist, taking into account the size of the nucleus.

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left|(2\pi)^2 m \langle \boldsymbol{k}'|V|\boldsymbol{k}\rangle\right|^2$$

["A femtometre — ten to the minus... five metres."]

$$V = -\frac{\cancel{Z} e^2}{r} = -Z e^2 \int \mathrm{d}\boldsymbol{x}' \frac{\rho(x')}{|\boldsymbol{x} - \boldsymbol{x}'|}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\bigg|_{\text{Rutherford}} \times F(q^2)$$

where $F(q^2)$ is the nuclear form factor.

$$F_{\text{nucl}}(q^2) = \int \mathrm{d}^3 x\, \rho(\boldsymbol{x})\, \mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{x}}$$

The electron won't excite the nucleus if it does not have relativistic energy, so $q \to 0$:

$$\simeq \int \mathrm{d}^3\boldsymbol{x}\, \rho(r) \left(1 + \mathrm{i}\, \boldsymbol{q}\cdot\boldsymbol{x} - \frac{1}{2}(\boldsymbol{q}\cdot\boldsymbol{x})^2 + \cdots\right) =$$

(We are assuming a symmetric charge distribution ρ in the nucleus. There are many nuclei where we would not want to do this.)

$$= 1 + 0 - \frac{1}{2} q_i\, q_j \underbrace{x^i x^j}_{= \frac{1}{3} r^2 \delta^{ij}} =$$

$$\left[\int \mathrm{d}\boldsymbol{x}\, \rho(r) x^i x^j = \frac{\delta^{ij}}{3} \int \mathrm{d}\boldsymbol{x}\, \rho(r)\, r^2 \right]$$

$$= 1 - \frac{1}{6} q^2 \underbrace{\int \mathrm{d}\boldsymbol{x}\, \rho(r)\, r^2}_{= \langle R^2 \rangle}$$

This enables us to find the size of the nucleus, by measuring $\mathrm{d}\sigma/\mathrm{d}\Omega$, calculate it thinking of the nucleus as a point, and divide to get the form factor, and thus $\langle R^2 \rangle$.

$A + \text{potential}$

$A + B \to A + B$

$A + B \to A + B^*$

$A + B \to C_1 + C_2 + \cdots + C_n$: is not within reach for nonrelativistic quantum mechanics. It is not just a technicality, it is conceptual. This reaction *require* relativity to proceed.

In non-relativistic quantum mechanic we study a $\psi(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$ governed by a Schrödinger equation

$$\left(-\frac{1}{2\, m} (\nabla_1^2 + \cdots + \nabla_n^2) + V(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) \right) \psi = E\, \psi$$

but this does not enable us to go from n particles to $n + 1$ particles (apart from the fact that the Schrödinger equation is not Lorentz invariant).

Many body quantum mechanics. Derive Hilbert space

$$\mathcal{J} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_n \oplus \cdots$$

where $\mathcal{H}_n$ is the Hilbert space for n particles.

Fock. $\mathcal{H}_0 \sim \mathbb{C}$. $|0\rangle$ vaccuum, $\langle 0|0\rangle = 1$. (Observe that the vacuum state $|0\rangle$ is not the zero state.)

Assume there is just one type of (spinless) particle in the Universe.

Creation/annihilation operators

$$\left[a_{\boldsymbol{k}_1}, a_{\boldsymbol{k}_2}^\dagger\right] = \delta^{(3)}(\boldsymbol{k} - \boldsymbol{k}')$$

$$[a, a] = \left[a^\dagger, a^\dagger\right] = 0$$

We want to define a one-particle state, which we write $|\boldsymbol{k}_1\rangle$, so tat $|\boldsymbol{k}_1\rangle = a_{\boldsymbol{k}_1}^\dagger |0\rangle$. I can go from an n particle state to an $n+1$ particle state:

$$a_{\boldsymbol{k}_{n+1}}^\dagger |\boldsymbol{k}_1, \ldots, \boldsymbol{k}_n\rangle = \mathfrak{Q}\, |\boldsymbol{k}_1, \ldots, \boldsymbol{k}_{n+1}\rangle.$$

The other definition that we need is that $a_{\boldsymbol{k}}$ kills the vaccuum: $a_{\boldsymbol{k}}|0\rangle = 0$. What does $a_{\boldsymbol{k}}$ do on a two-particle state?

$$a_{\boldsymbol{k}} |\boldsymbol{k}_1, \boldsymbol{k}_2\rangle = a_{\boldsymbol{k}}\, a_{\boldsymbol{k}_1}^\dagger\, a_{\boldsymbol{k}_2}^\dagger |0\rangle = \delta(\boldsymbol{k} - \boldsymbol{k}_1) \underbrace{\boldsymbol{a}_{k_2}^\dagger |0\rangle}_{|\boldsymbol{k}_2\rangle} + \delta(\boldsymbol{k} - \boldsymbol{k}_2) \underbrace{a_{k_1}^\dagger |0\rangle}_{= |\boldsymbol{k}_1\rangle}$$

$$\mathcal{H}_n \xrightarrow{a^\dagger} \mathcal{H}_{n+1}, \quad \mathcal{H}_n \xleftarrow{a} \mathcal{H}_{n+1}$$

A number operator N st. for any state $|\psi_m\rangle \in \mathcal{H}_n$, $N|\psi_n\rangle = n|\psi_n\rangle$.

$$N = \int \mathrm{d}^3\boldsymbol{k}\, a_{\boldsymbol{k}}^\dagger\, a_{\boldsymbol{k}}$$

We want $N|\boldsymbol{p}\rangle = 1|\boldsymbol{p}\rangle$. Let us make sure that it is:

$$\int \mathrm{d}^3\boldsymbol{k}\, a_{\boldsymbol{k}}^\dagger\, a_{\boldsymbol{k}}\, \underbrace{a_{\boldsymbol{p}}^\dagger |0\rangle}_{= |\boldsymbol{p}\rangle} = ?$$

We can replace $a_{\boldsymbol{k}} a_{\boldsymbol{p}}^\dagger |0\rangle$ by $\left[a_{\boldsymbol{k}}, a_{\boldsymbol{p}}^\dagger\right] |0\rangle = \delta(\boldsymbol{k} - \boldsymbol{p})|0\rangle$, since the extra term kills the vacuum.

$$\int \mathrm{d}^3\boldsymbol{k}\, a_{\boldsymbol{k}}^\dagger\, a_{\boldsymbol{k}}\, a_{\boldsymbol{p}}^\dagger |0\rangle = a_{\boldsymbol{p}}^\dagger |0\rangle = |\boldsymbol{p}\rangle$$

$$\underbrace{\int \mathrm{d}^3\boldsymbol{k}\;\boldsymbol{k}\, a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}}}_{= \boldsymbol{P}_{\text{tot}}} |\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n\rangle = \underbrace{(\boldsymbol{p}_1 + \cdots + \boldsymbol{p}_n)}_{= \boldsymbol{p}_{\text{tot}}} |\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n\rangle$$