2008–11–05

Next Monday at 10, homework at `http://fy.chalmers.se/~ferretti/`. The homework is to be handed in on Wednesday at eight. Mailbox at 6th floor, or email it to `ferretti@chalmers.se`.

Course evaluation. I need two people. Then all of you go to
`http://kursutv.portal.chalmers.se/ev.cgi?e=1429`

Partial waves

Rotationally invariant H $(V = V(r))$

$$f(\boldsymbol{k}, \boldsymbol{k}') = f(k, \theta) = \sum_{l=0}^{\infty} (2l+1)\, f_l(k)\, P_l(\cos\theta)$$

f is called the scattering amplitude, and $f_l(k)$ is the partial wave scattering amplitude.

$$\psi(\boldsymbol{x}) = \frac{1}{(2\pi)^{3/2}} \left(\mathrm{e}^{\mathrm{i}kz} + f(k, \theta)\, \frac{\mathrm{e}^{\mathrm{i}kr}}{r} \right)$$

Use

$$\mathrm{e}^{\mathrm{i}kz} = \sum_l (2l+1)\, P_l(\cos\theta)\, j_l(k\,r)\, \mathrm{i}^l$$

$$\mathrm{i}^l\, j_l(k\,r) \underset{kr\to\infty}{\longrightarrow} \frac{\mathrm{e}^{\mathrm{i}kr} - \mathrm{e}^{-\mathrm{i}(kr - l\pi)}}{2\,\mathrm{i}\,k\,r}$$

$$\psi(\boldsymbol{x}) = \frac{1}{(2\pi)^{3/2}} \left(\mathrm{e}^{\mathrm{i}kz} + f(k, \theta)\, \frac{\mathrm{e}^{\mathrm{i}kr}}{r} \right) =$$

$$= \frac{1}{(2\pi)^{3/2}} \sum_l \frac{(2l+1)}{2\,\mathrm{i}\,k} P_l(\cos\theta) \left((1 + 2\,\mathrm{i}\,k\, f_l(k)) \frac{\mathrm{e}^{\mathrm{i}kr}}{r} - \frac{\mathrm{e}^{-\mathrm{i}(kr - l\pi)}}{r} \right)$$

Conservation of probability $\Rightarrow$ flux in $=$ flux out.

$$1 = |1 + 2\,\mathrm{i}\,k\, f_l(k)|$$

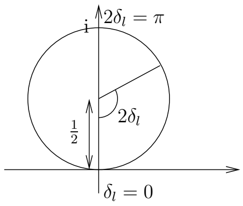
Argand Diagrams.

Definition

$$S_l(k) = \mathrm{e}^{2\mathrm{i}\delta_l(k)} = 1 + 2\,\mathrm{i}\,k\, f_l(k)$$

$$\Rightarrow k\, f_l = \frac{\mathrm{e}^{2\mathrm{i}\delta_l} - 1}{2\,\mathrm{i}}$$

$$f(k, \theta) = \sum_l (2\,l+1) \frac{1}{k} \mathrm{e}^{\mathrm{i}\delta_l} \sin\delta_l\, P_l(\cos\theta)$$



In the inelastic case the trajectory can spiral inwards in this diagram.

Optical theorem

$$\mathrm{Im}(f(k,\theta=0)) = \sum_l \frac{2\,l+1}{k}\sin^2\delta_l$$

$$\sigma_{\mathrm{tot}} = \int\ |f(k,\theta)|^2\,2\,\pi\,\mathrm{d}(\cos\theta) =$$

$$= \frac{1}{k^2}\cdot 2\pi \underbrace{\int_{-1}^{1}\mathrm{d}(\cos\theta)\sum_{l,l'} P_l(\cos\theta)\,P_{l'}(\cos\theta)}_{=2\delta_{l,l'}/(2l+1)} \times (2l+1)(2l'+1)\,\mathrm{e}^{\mathrm{i}\delta_l}\,\mathrm{e}^{-\mathrm{i}\delta_l}\sin\delta_l\sin\delta_{l'}$$

Obeys the optical theorem. (The eikonal approximation also obeys the optical theorem.)

Connection eikonal $\leftrightarrow$ Partial wave.

In the eikonal approximation $k\to\infty, l\to\infty, b=l/k$ fixed.

$$f(k,\theta) = \begin{cases} -\mathrm{i}k\int\ \mathrm{d}b\,b\,J_0(k\,b\,\theta)\left(\mathrm{e}^{2\mathrm{i}\Delta(b)}-1\right) \\ \sum \frac{2l+1}{k}\,P_l(\cos\theta)\,\frac{\mathrm{e}^{2\mathrm{i}\delta_l}-1}{2\,\mathrm{i}} \end{cases}$$

$J_0\sim P_l$ for small angles.

$$\delta_l(k) = \Delta\!\left(\frac{l}{k}\right)$$

Problem 6 Hard sphere s-wave scattering, with the partial wave method. The hard sphere has radius a.

$$V = \begin{cases} 0 & \text{if } r>a \\ \infty & \text{if } r<a \end{cases}$$

s-wave: $l=0$.

$$\psi = R_l(r)\,Y_{l\,m}(\theta,\varphi)$$

$$-\frac{1}{2\,m}\left(\frac{1}{r^2}\,\partial_r\,r^2\partial_r + \cancel{\frac{l(l+1)}{r^{w}}}\right)R = E\,R$$

$$R(r) = A\,\frac{\sin p\,r}{r} + B\,\frac{\cos p\,r}{r},\qquad E=\frac{p^2}{2m}$$

Boundary condition $R(a)=0\Rightarrow B/A=-\tan p\,a$.

$$R(r) = \left(\frac{A}{2\,\mathrm{i}}+\frac{B}{2}\right)\frac{\mathrm{e}^{\mathrm{i}pr}}{r} + \left(-\frac{A}{2\,\mathrm{i}}+\frac{B}{2}\right)\frac{\mathrm{e}^{-\mathrm{i}pr}}{r}$$

$$\underbrace{(1+2\,\mathrm{i}p\,f_{l=0}(k))}_{=\mathrm{e}^{2\mathrm{i}\delta_0}}\frac{\mathrm{e}^{\mathrm{i}pr}}{r} - \frac{\mathrm{e}^{-\mathrm{i}pr}}{r}$$

$$\Rightarrow \mathrm{e}^{2\mathrm{i}\delta_0} = \frac{\frac{A}{2\,\mathrm{i}}+\frac{B}{2}}{\frac{A}{2\,\mathrm{i}}-\frac{B}{2}} = \frac{-\mathrm{i}-\tan p\,a}{-\mathrm{i}+\tan p\,a}$$

$$\Rightarrow \delta_0 = -p\,a$$

$$\cos 2\delta_0 + \mathrm{i}\sin 2\,\delta_0 = \frac{1-\tan^2 p\,a - 2\,\mathrm{i}\tan p\,a}{1+\tan^2 p\,a}$$

$$\sin 2\delta_0 = -\frac{2\tan p\,a}{1+\tan^2 p\,a} = -2\cos p\,a\cdot\sin p\,a = \sin(-2\,p\,a)$$

$p\,a \ll 1$, low energy scattering. $l=0$ contributes.

$$\sigma_{\rm tot} = \sigma_{l=0} = \frac{4\pi}{p^2} \underbrace{\sin^2\delta_0}_{\approx p^2 a^2} = 4\,\pi\,a^2$$

Analytic properties of the S matrix. To us S is $\mathrm{e}^{2\mathrm{i}\delta_l(k)}$.

What happens if $k \in \mathbb{C}$?

Scattering wave

$$S_l(k) \quad \frac{\mathrm{e}^{\mathrm{i}kr}}{r} - \frac{\mathrm{e}^{-\mathrm{i}(kr - l\,n)}}{r}$$

Bound states:

$$\sim \frac{\mathrm{e}^{-\kappa r}}{r}$$

$$E_{\rm bound} = -\frac{\kappa^2}{2\,m}$$

Let $k = \mathrm{i}\,\kappa$.

$|S_l(k)| = 1$ for real k.

$S_l(k)$ has poles for $k = \mathrm{i}\kappa$ iff $\exists$ b.s. of angular momentum l and bind. energy $E_{\rm bs} = -\frac{\kappa^2}{2\,m}$.

Hard sphere

$$S_0 = \mathrm{e}^{2\mathrm{i}\delta_0} = \frac{-\mathrm{i} - \tan p\,a}{-\mathrm{i} + \tan p\,a}$$

poles: $\sin p\,a - \mathrm{i}\cos p\,a = 0$. $= -\mathrm{i}\cdot \mathrm{e}^{\mathrm{i}p\,a} \neq 0$.

Resonances:

Example

$$V = \begin{cases} -V_0, & r < a \\ 0, & r > a \end{cases}$$

$$V_{\rm eff}(r) = V(r) + \frac{l(l+1)}{r^2}$$

Metastable resonance states:

$$\frac{k^2}{2\,m} = E = E_{\rm res} - \mathrm{i}\,\frac{\Gamma}{2}$$

$$\mathrm{e}^{-\mathrm{i}Et} = \mathrm{e}^{-\mathrm{i}E_{\rm res} - \Gamma t/2}$$

decays.

$$S_l \approx \frac{\frac{k^2}{2'm} - E_{\rm res} - \frac{\mathrm{i}\,\Gamma}{2}}{\frac{k^2}{2\,m} - E_{\rm res} + \frac{\mathrm{i}\,\Gamma}{2}}$$