2008–11–04

$$\frac{1}{(2\pi)^{3/2}}\, f(\boldsymbol{p},\boldsymbol{p}') = -\frac{m}{2\pi}\int \mathrm{d}^3\boldsymbol{x}'\, \mathrm{e}^{-\mathrm{i}\boldsymbol{p}'\cdot\boldsymbol{x}'}\, V(\boldsymbol{x}')\, \psi(\boldsymbol{x}')$$

1st order Born approximation $\psi(\boldsymbol{x}') = \frac{1}{(2\pi)^{3/2}}\,\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} = \frac{1}{(2\pi)^{3/2}}\,\mathrm{e}^{\mathrm{i}pz}$.

The eikonal approximation (the semi-classical approximation).

$$\psi(\boldsymbol{x}) = \mathcal{N}(\boldsymbol{x})\,\mathrm{e}^{\mathrm{i}S(x)/\hbar}$$

In the end we look for things that survive as $\hbar \to 0$. The eikonal approximation will be valid for potentials that vary slowly compared to the wave function.

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi = E\psi$$

$$\nabla^2\psi = \nabla\cdot\nabla\left(\mathcal{N}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S}\right) = \nabla\cdot\left(\left(\nabla\mathcal{N} + \mathcal{N}\cdot\frac{\mathrm{i}}{\hbar}\nabla S\right)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S}\right) =$$

$$= \nabla\cdot\left(\nabla\mathcal{N} + \mathcal{N}\cdot\frac{\mathrm{i}}{\hbar}\nabla S\right)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S} + \left(\nabla\mathcal{N} + \mathcal{N}\cdot\frac{\mathrm{i}}{\hbar}\nabla S\right)\frac{\mathrm{i}}{\hbar}\,\nabla S\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S}$$

In the limit $\hbar \to 0$:

$$\frac{1}{2m}\mathcal{N}\,(\nabla S)^2\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S} + V\mathcal{N}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S} = E\mathcal{N}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S}$$

$$\frac{1}{2m}(\nabla S)^2 + V = E$$

Hamilton-Jacobi. $\mathcal{N}$ can be taken as the normalisation of the wave:

$$\mathcal{N} = \frac{1}{(2\pi)^{3/2}}$$

Consider a classical trajectory that is a straight line, situated a height b over the z axis. (The minimal distance between the incoming trajectory and the scattering centre: b is the impact parameter.) With the eikonal approximation we consider cases where the scattering angle is not very large. Here S is a function $S(\boldsymbol{b}, z)$.

$$\frac{1}{2m}\left(\frac{\mathrm{d}S}{\mathrm{d}z}\right)^2 + V(\boldsymbol{b}, z) = E$$

Often, V is taken to be a regular potential, a function of $\sqrt{b^2 + z^2}$.

$$\frac{\mathrm{d}S}{\mathrm{d}z} = \sqrt{2\,m\,(E - V)}$$

$$S(z) = \int_{z_0}^{z} \sqrt{p^2 - 2\,m\,V(\boldsymbol{b}, z')}\,\mathrm{d}z' + \text{constant} = [\text{putting constant} = p\,z_0] =$$

$$= \int_{z_0}^{z} \mathrm{d}z' \left(\sqrt{p^2 - 2\,m\,V(\boldsymbol{b}, z')} - p\right) + p\,(z - z_0) + p\,z_0 = \int_{-\infty}^{z} \mathrm{d}z' \left(\sqrt{p^2 - 2\,m\,V(\boldsymbol{b}, z')} - p\right) + p\,z =$$

$$= \int_{-\infty}^{z} \mathrm{d}z' \left(p\sqrt{1 - \frac{2\,m\,V}{p^2}} - p\right) + p\,z$$

$$p\sqrt{1 - \frac{2\,m\,V}{p^2}} - p \simeq p - \frac{m\,V}{p} - p = -\frac{m\,V}{p}$$

$$S \approx p\,z - \frac{m}{p}\int_{-\infty}^{z} V\left(\sqrt{b^2 + z'^2}\right)\mathrm{d}z'$$

$$\psi = \mathcal{N}\,\mathrm{e}^{\mathrm{i}S} = \frac{1}{(2\pi)^{3/2}} \cdot \mathrm{e}^{\mathrm{i}pz} \cdot \exp\left(-\mathrm{i}\frac{m}{p}\int_{-\infty}^{z} V\,\mathrm{d}z'\right)$$

$$f(\boldsymbol{p}, \boldsymbol{p}') = -\frac{m}{(2\pi)}\,(2\pi)^{3/2}\int \mathrm{d}\boldsymbol{x}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{p}'\cdot\boldsymbol{x}'}V(\boldsymbol{x}')\,\psi(\boldsymbol{x}')$$

$$\mathrm{d}^3\boldsymbol{x} = \mathrm{d}b \cdot b \cdot \mathrm{d}\varphi \cdot \mathrm{d}z'$$

$$f(\boldsymbol{p}, \boldsymbol{p}') = -\frac{m}{2\pi}\int \mathrm{d}b\,b\,\mathrm{d}\varphi\,\mathrm{d}z'\,\mathrm{e}^{-\mathrm{i}\boldsymbol{p}'\cdot\boldsymbol{x}'}V(\sqrt{b^2 + z^2})\,\mathrm{e}^{ip_1z}\exp\left(-\frac{\mathrm{i}\,m}{p}\int_{-\infty}^{z''} V(\sqrt{b^2 + z^2})\mathrm{d}z''\right)$$

$\boldsymbol{x}' = \boldsymbol{b} + z'\,\hat{\boldsymbol{z}}$:

$$f(\boldsymbol{p},\boldsymbol{p}') = -\frac{m}{2\pi}\int \mathrm{d}b\,b\,\mathrm{d}\varphi\,\mathrm{d}z'\,\mathrm{e}^{-\mathrm{i}\boldsymbol{p}'\cdot(\boldsymbol{b}+z'\hat{\boldsymbol{z}})}V(\sqrt{b^2+z^2})\,\mathrm{e}^{ip_1 z}\exp\!\left(-\frac{\mathrm{i}\,m}{p}\int_{-\infty}^{z''}V(\sqrt{b^2+z^2})\mathrm{d}z''\right)$$

We do the approximation for small scattering angles $\theta \approx 0$. $\boldsymbol{p}' = p\,(\cos\,\theta\;\hat{\boldsymbol{z}} + \sin\,\theta\;\hat{\boldsymbol{x}})$. $\boldsymbol{b} = b(\cos\varphi\,\hat{\boldsymbol{x}} + \sin\varphi\,\hat{\boldsymbol{y}})$. With this choice the integral works out like

$$-\frac{m}{2\pi}\int \mathrm{d}b\,b\,\mathrm{d}\varphi\,\mathrm{d}z'\exp(-\mathrm{i}\,b\,p\,\theta\cos\varphi)\,V(\sqrt{b^2+z^2})\exp\!\left(-\int V\right)$$

The integral over φ is a Bessel function:

$$\int \mathrm{d}\varphi\,\mathrm{e}^{-\mathrm{i}pb\theta\cos\varphi} = 2\pi\,J_0(p\,b\,\theta)$$

$$\int_{-\infty}^{+\infty}\mathrm{d}z'\,V\!\left(\sqrt{b^2+z'^2}\right)\cdot\exp\!\left(-\frac{\mathrm{i}\,m}{p}\int_{-\infty}^{z'}\mathrm{d}z''\,V\!\left(\sqrt{b^2+z''^2}\right)\right) =$$

The integrand is

$$\left[\frac{\mathrm{i}\,p}{m}\frac{\mathrm{d}}{\mathrm{d}z'}\exp\!\left(-\frac{\mathrm{i}\,m}{p}\int_{-\infty}^{z'}\mathrm{d}z''\,V\!\left(\sqrt{b^2+z''^2}\right)\right)\right]$$

$$= \frac{\mathrm{i}p}{m}\left(\exp\!\left(-\frac{\mathrm{i}\,m}{p}\int_{-\infty}^{\infty}\mathrm{d}z'\,V\!\left(\sqrt{b^2+z'}\right)\right) - 1\right)$$

$$f(\boldsymbol{p},\boldsymbol{p}') = -\,\mathrm{i}p\int_0^{\infty}\mathrm{d}b\,b\,J_0(p\,b\,\theta)\left(\mathrm{e}^{z\mathrm{i}\Delta(b)} - 1\right)$$

$$\Delta(b) := -\frac{m}{2p}\int_{-\infty}^{+\infty}V(\sqrt{b^2+z^2})\mathrm{d}z$$

Next Monday at 10, go to ⟨`http://fy.chalmers.se/~ferretti`⟩. Due on Wednesday at 8, put it in the mailbox at the 6th floor.

Now we do partial wave approximation.

Coordinate basis $|\boldsymbol{x}\rangle$.

$$X\,|\boldsymbol{x}\rangle = \boldsymbol{x}\,|\boldsymbol{x}\rangle$$

$$\langle\boldsymbol{x}\,|\boldsymbol{x}'\rangle = \delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}')$$

$\boldsymbol{X}$ corresponds to three commuting observables, X, Y, Z.

Momentum basis $|\boldsymbol{p}\rangle$. Three commuting observables $P_x = -\,\mathrm{i}\,\frac{\partial}{\partial x}, P_y, P_z$.

$$\boldsymbol{P}|\boldsymbol{p}\rangle = \boldsymbol{p}|\boldsymbol{p}\rangle$$

$$\langle \boldsymbol{x}|\boldsymbol{p}\rangle = \frac{1}{(2\pi)^{3/2}}\,\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}$$

Another set of commuting observables are $L_z, L^2, H_0 = -\,\frac{\nabla^2}{2m}$.

$$H_0|E,l,m\rangle = E|E,l,m\rangle$$

$$L^2|E,l,m\rangle = l\,(l+1)\,|E,l,m\rangle$$

$$L_z|E,l,m\rangle = m\,|E.l,m\rangle$$

$$\langle E',l',m'|E,l,m\rangle = \delta_{l,l'}\,\delta_{m,m'}\,\delta(E-E')$$

	$\lvert\boldsymbol{x}\rangle$	$\lvert\boldsymbol{p}\rangle$	$\lvert E,l,m\rangle$
$\langle\boldsymbol{x}'\rvert$	$\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}')$	$\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}'}$	
$\langle\boldsymbol{p}'\rvert$		$\delta(\boldsymbol{p}-\boldsymbol{p}')$	
$\langle E,l,m\rvert$			$\delta_{l,l'}\,\delta_{m,m'}\,\delta(E-E')$

$$\langle \boldsymbol{x}|E,l,m\rangle = \mathrm{i}^l\sqrt{\frac{2\,m\,p}{\pi}}\;j_l(p_r)Y_{lm}(\hat{\boldsymbol{r}})$$

$$\rightarrow -\frac{\nabla^2}{2\,m}\psi = -\frac{1}{2m}\left(\frac{1}{r^2}\,\partial_r r^2\partial_r + \frac{L^2}{r^2}\right)\psi$$

$$\psi = R(r)\cdot Y_{l,m}(\theta,\varphi), \quad L^2 Y_{l,m} = l\,(l+1)Y_{l,m}$$

$$-\frac{\nabla^2}{2\,m}\psi = -\frac{1}{2m}\left(\frac{1}{r^2}\partial_r\,r^2\,\partial_r R + \frac{l\,(l+1)}{r^2}R\right)Y_{lm} = E\,R\,Y_{lm}$$

$$\langle \boldsymbol{k}|E,l,m\rangle = \frac{1}{\sqrt{m\,k}}\,\delta\!\left(E - \frac{k^2}{2m}\right)Y_{lm}(\hat{\boldsymbol{k}})$$

$$\int \mathrm{d}\boldsymbol{x}\;\langle \boldsymbol{k}|\boldsymbol{x}\rangle\,\langle \boldsymbol{x}|E,l,m\rangle$$

$$f(\boldsymbol{k},\boldsymbol{k}') = -\,(2\pi)^2\,m\,\langle \boldsymbol{k}'|\underbrace{T|\boldsymbol{k}\rangle}_{=V|\psi\rangle} =$$

$$= -\,(2\pi)^2\,m\sum_{l,l',m,m'}\int\,\mathrm{d}E\,\mathrm{d}E'\,\langle \boldsymbol{k}'|E',l',m'\rangle\langle E',l',m'|T|E,l,m\rangle\langle E,l,m|\boldsymbol{k}\rangle =$$

Note

$$\langle E,l',m'|T|E,l,m\rangle = [\text{Wigner-Eckart}] = \delta_{ll'}\delta_{mm'}T_l(E)$$

$V = V(r)$

$$-(2\pi)^2 m \sum \int \mathrm{d}E\,\mathrm{d}E'\,\frac{1}{\sqrt{m\,k}} Y_{l'm'}\left(\hat{\boldsymbol{k}}'\right)\delta\left(\frac{k^2}{2\,m}-E'\right)\langle E',l',m'|T|E,l,m\rangle$$

$$\frac{1}{\sqrt{m\,k}}\,Y^*_{lm}(\hat{\boldsymbol{k}})\delta\left(\frac{k^2}{2\,m}-E\right)=$$

$|\boldsymbol{k}| = |\boldsymbol{k}'| = k.$

$$= -\frac{(2\pi)^2}{k}\sum_{l,m} T_l(E)\,Y_{lm}\left(\hat{\boldsymbol{k}}'\right)Y^*_{lm}\left(\hat{\boldsymbol{k}}\right)$$

Remember that $\hat{\boldsymbol{k}} = \hat{\boldsymbol{z}}\,, \boldsymbol{k} = k\,\hat{\boldsymbol{z}}$.

$$Y^*_{lm}(\hat{\boldsymbol{z}}) = \sqrt{\frac{2\,l+1}{4\pi}}\,\delta_{m\,0}$$

$$= -\frac{(2\pi)^2}{k}\sum_{l} T_l(E)\sqrt{\frac{2\,l+1}{4\pi}}\,Y_{l0}\left(\hat{\boldsymbol{k}}'\right)$$

$$Y_{l,0}\left(\hat{\boldsymbol{k}}'\right) = \sqrt{\frac{2\,l+1}{4\pi}}\,P_l(\cos\theta)$$

$$= -\frac{\pi}{k}\sum_{l}\,(2\,l+1)\cdot T_l\left(\frac{k^2}{2m}\right)P_l(\cos\theta)$$

Definition

$$f_l(k) = -\frac{\pi}{k}\,T_l\left(\frac{k^2}{2\,m}\right)$$

Partial wave f amplitude

$$f(k,\theta) = \sum_{l=0}^{\infty}\,(2l+1)\,f_l(k)\,P_l(\cos\theta)$$

This is our starting point.