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$$H = H_0 + V$$

$$H_0|\boldsymbol{p}\rangle = \frac{\boldsymbol{p}^2}{2\,m}|\boldsymbol{p}\rangle$$

$$H|\psi\rangle = E|\psi\rangle$$

Lippmann-Schwinger equation:

$$|\psi\rangle = |\boldsymbol{p}\rangle + \frac{1}{E - H_0 + \mathrm{i}\varepsilon} V|\psi\rangle$$

We will see, in a minute, what goes wrong when we choose the negative sign, $-\mathrm{i}\varepsilon$, above.

This can be written more explicitly:

$$\psi(\boldsymbol{x}) = \frac{1}{(2\pi)^{3/2}} \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} - 2\,m \int \mathrm{d}\boldsymbol{x}'\, G(\boldsymbol{x},\boldsymbol{x}') \frac{\mathrm{e}^{\mathrm{i}p|\boldsymbol{x}-\boldsymbol{x}'|}}{4\pi\,|\boldsymbol{x}-\boldsymbol{x}'|} V(\boldsymbol{x}')\psi(\boldsymbol{x}') \tag{1}$$

V will be nonzero only in a fairly restricted area, and the detector would typically be many orders of magnitude away. We want the wave function where the detector is, and eventually get the cross section. Thus, within this integral, we assume $|\boldsymbol{x}'| \ll |\boldsymbol{x}| = r$.

$$|\boldsymbol{x}-\boldsymbol{x}'| = \sqrt{\boldsymbol{x}^2 + \boldsymbol{x}'^2 - 2\,\boldsymbol{x}\cdot\boldsymbol{x}'} \simeq r\sqrt{1 - 2\frac{\boldsymbol{x}\cdot\boldsymbol{x}'}{r^2}} \simeq r\left(1 - \frac{\boldsymbol{x}\cdot\boldsymbol{x}'}{r^2}\right) \simeq r - \frac{\boldsymbol{x}\cdot\boldsymbol{x}'}{r} = r - \hat{\boldsymbol{x}}\cdot\boldsymbol{x}'$$

where we have defined the unit vector $\hat{\boldsymbol{x}} = \boldsymbol{x}/r$.

From 1:

$$\psi(\boldsymbol{x}) = \frac{1}{(2\pi)^{3/2}} \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} - \frac{m}{2\pi}\cdot\frac{\mathrm{e}^{\mathrm{i}pr}}{r}\cdot\int \mathrm{d}\boldsymbol{x}'\,\mathrm{e}^{\mathrm{i}p\hat{\boldsymbol{x}}\cdot\boldsymbol{x}'} V(\boldsymbol{x}')\psi(\boldsymbol{x}')$$

Now I introduce a new quantity, a vector $\boldsymbol{p}' := p\hat{\boldsymbol{x}}$:

$$\psi(\boldsymbol{x}) = \frac{1}{(2\pi)^{3/2}} \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} - \frac{m}{2\pi}\cdot\frac{\mathrm{e}^{\mathrm{i}pr}}{r}\cdot\int \mathrm{d}\boldsymbol{x}'\,\mathrm{e}^{\mathrm{i}\boldsymbol{p}'\cdot\boldsymbol{x}'} V(\boldsymbol{x}')\psi(\boldsymbol{x}')$$

So the particle comes in with momentum $\boldsymbol{p}$ and is scattered with a momentum $\boldsymbol{p}'$, the angle between $\boldsymbol{p}$ and $\boldsymbol{p}'$ being θ.

Definition: The scattering amplitude $f(\boldsymbol{p},\boldsymbol{p}')$:

$$\frac{1}{(2\pi)^{3/2}} f(\boldsymbol{p},\boldsymbol{p}') := \frac{m}{2\pi}\int \mathrm{d}\boldsymbol{x}'\cdot\mathrm{e}^{-\mathrm{i}\boldsymbol{p}'\cdot\boldsymbol{x}'} V(\boldsymbol{x}')\,\psi(\boldsymbol{x}').$$

$$\psi(\boldsymbol{x}) = \frac{1}{(2\pi)^{3/2}}\left(\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} + \frac{\mathrm{e}^{\mathrm{i}pr}}{r} f(\boldsymbol{p},\boldsymbol{p}')\right) \tag{2}$$

$\boldsymbol{p}$ depends on ψ, so we still have an implicit equation.

$$f(\boldsymbol{p},\boldsymbol{p}') = f(E,\theta), \quad E = \frac{\boldsymbol{p}^2}{2\,m}$$

We normaly consider only rotationally symmetric system; if not we have to add the azimuthal angle to the list of variables. f looks like it depends on two vectors, but it really depends on an angle.

$$f(\boldsymbol{p},\boldsymbol{p}')=-(2\pi)^2\cdot m\,\langle\boldsymbol{p}'|V|\psi\rangle$$

This is another form of the scattering amplitude you see often. It is still implicit, in how it depends on ψ.

We will show the following simple formula:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}=|f(\theta)|^2$$

Knowing this we only need to find an approximation of f.

(Some of the normalization constants seem a bit awkward. They are not all that important. Things get easier in relativistic quantum mechanics.)

With a $-\mathrm{i}\varepsilon$ we would get

$$\psi^{(-)}(\boldsymbol{x})=\frac{1}{(2\pi)^{3/2}}\left(\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}+\frac{\mathrm{e}^{-\mathrm{i}pr}}{r}f(\boldsymbol{p},\boldsymbol{p}')\right)$$

with $-p$ instead of $+p$. What is the physical significance of this?

With a plus, $\mathrm{e}^{+\mathrm{i}pr}/r$ is an outgoing radial wave (with some normalization). With a minus, $\mathrm{e}^{-\mathrm{i}pr}/r$ is an *incoming* radial wave.

Taking the plus sign now, how do we get to the cross section?

The current associated with a wave function

$$\boldsymbol{J}=\frac{1}{2\,\mathrm{i}\,m}(\psi^*\nabla\psi-\psi\nabla\psi^*),\quad\nabla\cdot\boldsymbol{J}=0$$

This is the trick we use to avoid using wave packets. The expression for ψ above (2) consists of two parts: first the incoming plane wave and then the scattered wave.

$$\boldsymbol{J}_{\mathrm{in}}=\frac{1}{2\,\mathrm{i}\,m}\left(\frac{1}{(2\pi)^{3/2}}\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}\cdot\nabla\frac{1}{(2\pi)^{3/2}}\mathrm{e}^{+\mathrm{i}\boldsymbol{p}'\cdot\boldsymbol{x}'}+\text{complex conjugate}\right)=$$

$$=\frac{1}{2\mathrm{i}\,m}\left(\frac{1}{(2\,m)^3}\,\mathrm{i}\,\boldsymbol{p}+\text{complex conjugate}\right)=\frac{1}{(2\pi)^3}\frac{\boldsymbol{p}}{m}=\frac{1}{(2\pi)^3}\boldsymbol{v}$$

We are interested in the radial part, J_r, of the outgoing wave, since we want the flux through a detector taking up a certain solid angle.

$$J_r^{\mathrm{out}}=\frac{1}{2\,\mathrm{i}\,m}\left(\frac{1}{(2\pi)^{3/2}}\cdot\frac{\mathrm{e}^{-\mathrm{i}pr}}{r}f^*\cdot\frac{\partial}{\partial r}\left(\frac{1}{(2\pi)^{3/2}}\frac{\mathrm{e}^{\mathrm{i}pr}}{r}f\right)-\text{complex conjugate}\right)=$$

$$=\frac{1}{2\,\mathrm{i}\,m}\left(\frac{1}{(2\pi)^3}|f|^2\frac{\mathrm{e}^{-\mathrm{i}pr}}{r}\left(\frac{\mathrm{i}p\,\mathrm{e}^{\mathrm{i}pr}}{r}-\frac{\mathrm{e}^{\mathrm{i}pr}}{r^2}\right)-\text{complex conjugate}\right)=$$

Non-leading terms can be dropped [or is it taken out by the complex conjugate?], and we get:

$$=\frac{|f|^2}{(2\pi)^3}\cdot\frac{p}{m}\cdot\frac{1}{r^2}$$

We had $\dot{N} = \Phi\,\sigma$, the number of particles through an area $\mathrm{d}A = r^2\mathrm{d}\Omega$ equals

$$\#\mathrm{part} = J_r\, r^2\, \mathrm{d}\Omega = \frac{|f|^2}{(2\pi)^3} \cdot \frac{p}{m} \cdot \frac{1}{r^2} \cdot r^2 \cdot \mathrm{d}\Omega = \frac{1}{(2\pi)^3} \cdot \frac{p}{m}\,\mathrm{d}\sigma$$

Thus the differential cross section is given by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = |f(\theta)|^2$$

$\mathrm{d}\Omega = |\mathrm{d}(\cos\theta)\,\mathrm{d}\varphi|$.

$$\text{Total cross section}\, \sigma_{\mathrm{tot}} = \int \mathrm{d}\Omega \cdot \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$$

$$f(\boldsymbol{p}, \boldsymbol{p}') = -\,(2\pi)^2\, m\, \langle \boldsymbol{p}' | V | \psi \rangle$$

The three approximations we will consider are:

– The Born approximation.

– The eikonal approximation.

– The partial wave approximation (it is not strictly an approximation, but we will use the method to do approximations).

There is one more formal aspect that we need to speek of too, the optical theorem. We will, however, treat the Born approximation first.

We started out with the exact equation

$$|\psi\rangle = |\boldsymbol{p}\rangle + \frac{1}{E - H_0 + \mathrm{i}\varepsilon}\, V\, |\psi\rangle$$

This is the first step towards an approximation; if V is small we get just the incoming wave:

$$|\psi^{(0)}\rangle = |\boldsymbol{p}\rangle$$

$$|\psi^{(1)}\rangle = |\boldsymbol{p}\rangle + \frac{1}{E - H_0 + \mathrm{i}\varepsilon}\, V\, \underbrace{|\boldsymbol{p}\rangle}_{= |\psi^{(0)}\rangle}$$

We can get a recursion relation for $|\psi\rangle$ to the n:th order:

$$|\psi^{(n)}\rangle = |\boldsymbol{p}\rangle + \frac{1}{E - H_0 + \mathrm{i}\varepsilon} |\psi^{(n-1)}\rangle.$$

We are intrested in the first order Born approximation: everywhere you see $|\psi\rangle$, replace it with $|\boldsymbol{p}\rangle$.

$$f^{(1)}(\boldsymbol{p}, \boldsymbol{p}') = -\frac{m}{2\pi} \int \mathrm{d}\boldsymbol{x}' \cdot \mathrm{e}^{\mathrm{i}(\boldsymbol{p} - \boldsymbol{p}')\cdot \boldsymbol{x}'}\, V(\boldsymbol{x}')$$

Example: $V(\boldsymbol{x}') = V(r), r = |\boldsymbol{x}'|$.

$$f^{(1)} = -\frac{m}{2\pi} \int \mathrm{d}\varphi\, \mathrm{d}(\cos\theta')\, r^2 \mathrm{d}r\, \mathrm{e}^{\mathrm{i}|\boldsymbol{p} - \boldsymbol{p}'| r \cos\theta'}\, V(r)$$

The integral over φ gives a factor 2π, cancelling the 2π in the denominator.

$$f^{(1)} = -\,m\int_0^{+\infty} \mathrm{d}r\, r^2\, \frac{\mathrm{e}^{+\mathrm{i}|\boldsymbol{p}-\boldsymbol{p}'|r} - \mathrm{e}^{-\mathrm{i}|\boldsymbol{p}-\boldsymbol{p}'|r}}{\mathrm{i}\,|\boldsymbol{p}-\boldsymbol{p}'|}\, V(r)$$

For a central potential the problem is reduced to an integral over one variable. To make things a bit more explicit:

$$|\boldsymbol{p}-\boldsymbol{p}'|^2 = \boldsymbol{p}^2 + \boldsymbol{p}'^2 - 2\,\boldsymbol{p}\cdot\boldsymbol{p}'$$

$\boldsymbol{p}'$ has the same length as $\boldsymbol{p}$, but is pointing towards the detector.

$$|\boldsymbol{p}-\boldsymbol{p}'|^2 = p^2 + p^2 - 2\,p^2\cos\theta = 2p^2(1-\cos\theta) = 4\,p^2\sin^2\frac{\theta}{2}$$

(This θ is the vector between $\boldsymbol{p}$ and $\boldsymbol{p}'$, the incoming direction and the direction towards the detector.)

$$\Rightarrow |\boldsymbol{p}-\boldsymbol{p}'| = 2\,p\sin\frac{\theta}{2}$$

Whether the integral can be done, naturally depends on V. We will now consider one V for which this is possible: The Yukawa potential.

$$V = \frac{V_0}{\mu}\cdot\frac{\mathrm{e}^{-\mu r}}{r}$$

(The V_0/r is just a normalisation that the book has chosen.) This is a short range potental, that decays rapidly beyond $r\sim 1/\mu$. Yukawa proposed this potential for nuclear interaction, with μ being the mass of the π meson, $\sim 140\,\mathrm{MeV}$. The m we had above is not the mass of an interaction particle, but the (reduced) mass of e.g. a proton being scattered, $\sim 980\,\mathrm{MeV}$.

$$f_{\text{Yuk}}^{(1)} = \ldots$$

It is absolutely trivial. Everybody must be able to do this integral.

$$f_{\text{Yuk}}^{(1)} = -\frac{m}{\mathrm{i}\,|\boldsymbol{p}-\boldsymbol{p}'|}\cdot\frac{V_0}{\mu}\int_0^{\infty}\mathrm{d}r\, r\cdot\frac{1}{r}\left(\mathrm{e}^{\mathrm{i}|\boldsymbol{p}-\boldsymbol{p}'|r-\mu r} - \text{complex conjugate}\right) = \ldots =$$

$$= -\frac{2\,m\,V_0}{\mu}\cdot\frac{1}{\mu^2 + |\boldsymbol{p}-\boldsymbol{p}'|^2} = -\frac{2\,m\,V_0}{\mu}\cdot\frac{1}{\mu^2 + 4p^2\sin^2\frac{\theta}{2}}.$$

Question: In what reference frame are we? In the centre of mass frame. If we want to be in a laboratory frame $\neq$ the centre of mass frame, we get a different θ.

Now, let us forget Yukawa, and make the following argument. Let $\mu\to 0$ keeping V_0/μ fixed.

$$\frac{V_0}{\mu} = Z\,Z'\,e^2, \quad V\to\frac{Z\,Z'\,e^2}{r}\ \text{as}\ \mu\to 0$$

This V is the Coulomb potential for two ions. This is the most famous cross section you will ever see:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\to\left(-\,2\,m\,Z\,Z'\,e^2\cdot\frac{1}{4p^2\sin^2\frac{\theta}{2}}\right)^2$$

This is the classical result. The first order Born approximation is classical physics. Not always, but in this case it is. All you need to do in quantum mechanics is to evaluate an integral. Using $f^{(2)}$ and restoring $\hbar$:s, we would find $\hbar$:s there, which we do not get when doing $f^{(1)}$ (from dimensional analysis, then $\hbar \to 0$ gives classical physics).

Going from the centre of mass system to the laboratory frame

Let us do the general case of an elastic scattering of two particles with different masses m_1 and m_2, interacting with an arbitrary potential $V(\boldsymbol{r}_1 - \boldsymbol{r}_2)$, not necessarily spherically symetric. Define the relative coordinate $\boldsymbol{r} = \boldsymbol{r}_2 - \boldsymbol{r}_1$. The centre of mass coordinate is

$$\boldsymbol{R} = \frac{m_1 \boldsymbol{r}_1 + m_2 \boldsymbol{r}_2}{m_1 m_2}$$

$\mu = m_1 m_2 / (m_1 + m_2)$ is the reduced mass. $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are the coordinates in *the laboratory frame*. $\boldsymbol{r}_1'$ and $\boldsymbol{r}_2'$ are the coordinates in *the centre of mass frame*, defined by $\boldsymbol{R} = \boldsymbol{0}$.

$$\boldsymbol{r}_1' = -\frac{m_2}{m_1 + m_2} \boldsymbol{r} = -\frac{\mu}{m_1} \boldsymbol{r}$$

$$\boldsymbol{r}_2' = +\frac{m_2}{m_1 + m_2} \boldsymbol{r} = +\frac{\mu}{m_2} \boldsymbol{r}$$