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Sakurai, Chapter 7, Scattering.

Relativistic quantum mechanics (Peskin & Schroeder, Introduction to Quantum Field Theory).

Scattering is a fancy word for "collision". We want to calculate the *cross section*. So what is a cross section?

A classical example: You have a certain flow, of air for instance. You have a flux: the number of particles through a certain area per time: $\Phi = \#\text{part}/(\text{area} \times \text{time})$.

$$N = \Phi \cdot A \cdot t, \quad [\Phi] = \text{cm}^{-2} \cdot \text{s}^{-1}$$

Suppose there is a bottle immersed in the flux. The number of particles that hit the bottle in unit time $\dot{N}_{\text{hits}} = \Phi\, \sigma$, where σ is a number with dimension of area. σ here, is the geometrical cross section. The concept of cross section generalises to more or less anything. You just have to generalise what you mean by "hit".

An "*event*" is something that happens. $A + B \to C_1 + \cdots + C_n$. Is X produced by this reaction, if yes, we call that an event. $A + B \to A + B$. Is particle A going to hit my detector, that takes up a certain solid angle? That is an event. These events are all proportional to the flux. The proportionality constant is the cross section for that event.

LHC, colliders: $\dot{N} = L \cdot \sigma$, where L is called the luminosity (and is... the flux). σ is the total cross section.

Differential cross section Suppose I have particle A and particle B colliding. I take a tiny little detector, with a small solid angle $\Delta\Omega$.

$$\Delta\dot{N} = \Phi\, \underbrace{\Delta\Omega \cdot \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}}_{=\Delta\sigma}$$

The number of events has to be proportional to the flux, and also, to first approximation and exactly in the infinitesimal approximation, to the small solid angle $\Delta\Omega$. $\mathrm{d}\sigma/\mathrm{d}\Omega$ becomes a kind of cross section density.

["What date is it? Is it Tuesday? My watch says Monday. It is a wonder that I didn't miss the class" — Gabriele Ferretti]

The Lippmann-Schwinger Equation is an exact (but implicit ["If I say it is useless, my colleagues get upset"]) equation for σ. Approximations will give us an explicit equation: Born approximation, Eikonal approximation, Partial wave approximation.

Let us consider scattering of a particle from a potential. If I can solve this problem, I can also solve the problem of the elastic scattering of two particles.

$$H = \frac{\boldsymbol{p}_A^2}{2m_A} + \frac{\boldsymbol{p}_B^2}{2\,m_B} + V(\boldsymbol{x}_A - \boldsymbol{x}_B)$$

To get the problem of scattering of a particle from a potential, I go to the centre of mass frame. $\boldsymbol{r} = \boldsymbol{x}_A - \boldsymbol{x}_B$, $\boldsymbol{R} = (m_A\, \boldsymbol{x}_A + m_B\, \boldsymbol{x}_B)/(m_A + m_B)$. For $\boldsymbol{r}$ we have a momentum $\boldsymbol{p}$ and reduced mass $m = m_A m_B/(m_A + m_B)$, for $\boldsymbol{R}$ we get a momentum $\boldsymbol{P}$ and the Hamiltonian of a free particle, which we ignore. Thus $H = \boldsymbol{p}^2/2\,m + V(\boldsymbol{r})$.

We start in the far past, $t = -\infty$, and consider a wave packet moving towards the (localised) potential V. At the end of the day, $t = +\infty$, I will have a really messy situation with the wave-function spread all over the place. We ask the question: How will the wave-function look at $t = +\infty$.

There is an extra embedded complication. Wave packets are hell. We are going to do something that is physically wrong, and consider plane waves. If we know what happens with all the plane waves, we know everything about the wave packets too.

The plane wave is everywhere (including where it shouldn't be, behind the potential). We now turn to the Lippmann-Schwinger theory. We are going to do perturbation theory. (Here we are in a continuum, so we can't study how energy levels change under influence of a perturbing potential, since there are no energy levels. Therefore, we keep the energy fixed, and look at how the wave function changes.)

$$H = H_0 + V(\boldsymbol{x})$$

where $H_0 = \boldsymbol{p}^2/2\,m$. (We set $\hbar = 1$ throughout the course. $c = 1$ too, but since we are not doing relativity that doesn't matter much here.)

$$H_0 = \frac{\boldsymbol{p}^2}{2m} = -\frac{\nabla^2}{2m}$$

That is our free Hamiltonian.

$$H_0|\boldsymbol{p}\rangle = \frac{\boldsymbol{p}^2}{2\,m}\,|\boldsymbol{p}\rangle$$

$$\langle \boldsymbol{x}|\boldsymbol{p}\rangle = \phi_{\boldsymbol{p}}(\boldsymbol{x}) = \frac{1}{(2\pi)^{3/2}}\exp(\mathrm{i}\,\boldsymbol{p}\cdot\boldsymbol{x})$$

This is the free equation. We want to find the solution to $H|\psi\rangle = E|\psi\rangle$, with the same E. There is no meaning in perturbing the energy, but $|\psi\rangle$ will be perturbed, $|\psi\rangle = |\boldsymbol{p}\rangle + \text{something}$. For that something the Lippmann-Schwinger equation gives an exact equation... that depends on $|\psi\rangle$.

$$|\psi\rangle = |\boldsymbol{p}\rangle + \frac{1}{E - H_0}\,V\,|\psi\rangle$$

This is nonsense! Why is this nonsense? It is not nonsense because we take the inverse of a differential operator. If E did not belong to the spectrum of H_0 this would be perfectly fine. But we have $(E - H_0)|\psi\rangle = 0$. We will fix this in a moment. First, act with $(E - H_0)$ on both sides:

$$(E - H_0)|\psi\rangle = \underbrace{(E - H_0)\,|\boldsymbol{p}\rangle}_{=0\,\text{by construction}} + (E - H_0)\frac{1}{E - H_0}V|\psi\rangle \quad \Rightarrow \quad H|\psi\rangle = E|\psi\rangle$$

The problem is that E belongs to the spectrum. Let us take the energy a little bit of the real line:

$$|\psi\rangle = |\boldsymbol{p}\rangle + \frac{1}{E - H_0 + \mathrm{i}\,\varepsilon}\,V\,|\psi\rangle$$

It would not exactly cancel as above, when we act with $(E - H_0)$ from the left, but it works out when we take the limit $\varepsilon \to 0$. There is an ambiguity: It could be $+\mathrm{i}\,\varepsilon$ or $-\mathrm{i}\varepsilon$, and they are different. We will do it with the plus, and the calculation will show what would go wrong if we were to choose the minus sign. [Sakurai: "This is known as the Lippmann-Schwinger equation" and "The Lippmann-Schwinger equation is a *ket equation* independent of particular representations." It seems to me, Christian, that Gabriele may mean something more general, or else, the Lippmann-Schwinger equation in the coordinate representation that we now derive.]

Let us now consider the same thing in the coordinate representation:

$$\langle \boldsymbol{x}|\psi\rangle = \langle \boldsymbol{x}|\boldsymbol{p}\rangle + \langle \boldsymbol{x}|\frac{1}{E - H_0 + \mathrm{i}\varepsilon} V|\psi\rangle$$

$$\langle \boldsymbol{x}|\psi\rangle = \psi(\boldsymbol{x}); \quad \langle \boldsymbol{x}|\boldsymbol{p}\rangle = \frac{1}{(2\,\pi)^{3/2}}\,\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}$$

The rightmost term looks horrible, but it is not. We separate the two operators, by introducing a complete set of states, say, eigenstates of position:

$$\langle \boldsymbol{x}|\frac{1}{E - H_0 + \mathrm{i}\varepsilon} V|\psi\rangle = \int \mathrm{d}^3\boldsymbol{x}'\,\langle \boldsymbol{x}|\frac{1}{E - H_0 + \mathrm{i}\varepsilon}|\boldsymbol{x}'\rangle\langle \boldsymbol{x}'|V|\psi\rangle.$$

(Dirac is a notational genius)

$$\langle \boldsymbol{x}|\frac{1}{E - H_0 + \mathrm{i}\varepsilon}|\boldsymbol{x}'\rangle = 2m\,G_E(\boldsymbol{x}, \boldsymbol{x}') = \text{some function, that we will compute in a second.}$$

$$\langle \boldsymbol{x}'|V|\psi\rangle = V(\boldsymbol{x}')\,\psi(\boldsymbol{x}')$$

$$G_E(\boldsymbol{x}, \boldsymbol{x}') = \frac{1}{2\,m}\,\langle \boldsymbol{x}|\frac{1}{E - H_0 + \mathrm{i}\varepsilon}|\boldsymbol{x}'\rangle$$

In the position representation, $H_0 = -\frac{\nabla^2}{2\,m}$. It is far easier in the momentum representation, $H_0 = \frac{\boldsymbol{p}^2}{2\,m}$. Let us put in a complete set of states:

$$G_E(\boldsymbol{x}, \boldsymbol{x}') = \frac{1}{2\,m}\int \mathrm{d}^3\boldsymbol{q}\,\mathrm{d}^3\boldsymbol{k}\;\langle \boldsymbol{x}|\boldsymbol{q}\rangle\langle \boldsymbol{q}|\frac{1}{E - H_0 + \mathrm{i}\,\varepsilon}|\boldsymbol{k}\rangle\langle \boldsymbol{k}|\boldsymbol{x}'\rangle$$

$$\langle \boldsymbol{x}|\boldsymbol{q}\rangle = \frac{1}{(2\pi)^{3/2}}\,\mathrm{e}^{\mathrm{i}\boldsymbol{x}\cdot\boldsymbol{q}}, \quad \langle \boldsymbol{k}|\boldsymbol{x}'\rangle = \frac{1}{(2\pi)^{3/2}}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$$

$$\langle \boldsymbol{q}|\frac{1}{E - H_0 + \mathrm{i}\,\varepsilon}|\boldsymbol{k}\rangle = \frac{1}{E - \frac{\boldsymbol{k}^2}{2\,m} + \mathrm{i}\,\varepsilon}\,\delta^{(3)}(\boldsymbol{q} - \boldsymbol{k})$$

$$G_E(\boldsymbol{x}, \boldsymbol{x}') = \frac{1}{2\,m}\int \frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3}\cdot\exp(\mathrm{i}\,\boldsymbol{q}\cdot(\boldsymbol{x} - \boldsymbol{x}'))\cdot\frac{1}{\frac{\boldsymbol{p}^2}{2\,m} - \frac{\boldsymbol{q}^2}{2\,m} + \mathrm{i}\,\varepsilon}$$

$\varepsilon \to 0 \Rightarrow 2\,m\,\varepsilon \to 0$. We change the definition of ε as we go along:

$$G_E(\boldsymbol{x}, \boldsymbol{x}') = \int \frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3}\exp(\mathrm{i}\,\boldsymbol{q}\cdot(\boldsymbol{x} - \boldsymbol{x}'))\cdot\frac{1}{p^2 - q^2 + \mathrm{i}\,\varepsilon} =$$

Now, we go to polar coordinates in q. [Methinks this is spherical coordinates (q, θ, φ), where the direction $\boldsymbol{x} - \boldsymbol{x}'$ taken as the zenith, i.e., the direction of reference for the zenith angle θ, with φ being the azimuthal angle as usual. At first, I thougth Gabriele was making a sign error here. The volume element in spherical coordinates is $\mathrm{d}V = q^2 \sin\theta\,\mathrm{d}q\,\mathrm{d}\theta\,\mathrm{d}\varphi$, and with $\mathrm{d}(\cos\theta) = -\sin\theta\,\mathrm{d}\theta$ we would get $\mathrm{d}V = -q^2\,\mathrm{d}q\,\mathrm{d}(\cos\theta)\,\mathrm{d}\varphi$. But we integrate $\cos\theta$ from -1 to 1, corresponding to θ from π to 0, rather than 0 to π as we would normally do. The interchange of the integration limits changes the sign, giving $\mathrm{d}V = q^2\,\mathrm{d}q\,\mathrm{d}(\cos\theta)\,\mathrm{d}\varphi$.]

$$= \int \frac{q^2\cdot\mathrm{d}q\cdot\mathrm{d}(\cos\theta)\cdot\mathrm{d}\varphi}{(2\pi^3)}\,\mathrm{e}^{\mathrm{i}\,q\,|\boldsymbol{x} - \boldsymbol{x}'|\cos\theta}\,\frac{1}{p^2 - q^2 + \mathrm{i}\varepsilon} =$$

Integrating over φ gives a factor 2π:

$$= \int_0^\infty \frac{q^2\,\mathrm{d}q}{(2\pi)^2} \cdot \underbrace{\frac{1}{\mathrm{i}\,q\,|\boldsymbol{x}-\boldsymbol{x}'|}\left(\mathrm{e}^{\mathrm{i}q|\boldsymbol{x}-\boldsymbol{x}'|} - \mathrm{e}^{-\mathrm{i}q|\boldsymbol{x}-\boldsymbol{x}'|}\right)}_{=\int_{-1}^{1}\mathrm{d}u\,\exp(\mathrm{i}q|\boldsymbol{x}-\boldsymbol{x}'|u)} \cdot \frac{1}{p^2 - q^2 + \mathrm{i}\,\varepsilon} =$$

We change the integration limit, so that we only need one term in the exponential factor.

$$= \frac{1}{4\pi^2} \cdot \frac{1}{\mathrm{i}\,|\boldsymbol{x}-\boldsymbol{x}'|} \int_{-\infty}^{+\infty} q\,\mathrm{d}q\,\mathrm{e}^{\mathrm{i}q|\boldsymbol{x}-\boldsymbol{x}'|} \cdot \frac{1}{p^2 - q^2 + \mathrm{i}\varepsilon} =$$

$$= \frac{1}{4\pi^2} \cdot \frac{\mathrm{i}}{|\boldsymbol{x}-\boldsymbol{x}'|} \int_{-\infty}^{+\infty} q\,\mathrm{d}q\,\mathrm{e}^{\mathrm{i}q|\boldsymbol{x}-\boldsymbol{x}'|} \cdot \frac{1}{q^2 - p^2 - \mathrm{i}\varepsilon} =$$

We will use the residue theorem. We think of the integration variable q as belonging to the complex plane, and we want to integrate it from $-\infty$ to $+\infty$. If it weren't for the ε, we would have two poles at $q = \pm p$. Now we have $q^2 - p^2 - \mathrm{i}\varepsilon = (q - p - \mathrm{i}\varepsilon')(q + p + \mathrm{i}\varepsilon') = q^2 - p^2 + \mathrm{i}\,\varepsilon'\,(q - p) - \mathrm{i}\,\varepsilon'(q + p) + \varepsilon'^2 \simeq q^2 - p^2 - 2\,\mathrm{i}\,\varepsilon'\,p$.

I have to close the integration contour upstairs, enclosing the pole $p + \mathrm{i}\varepsilon$ and leaving out $-p - \mathrm{i}\varepsilon$; otherwise we would have a divergence.

$$= \frac{1}{4\pi^2} \cdot \frac{\mathrm{i}}{|\boldsymbol{x}-\boldsymbol{x}'|} \cdot 2\,\pi\,\mathrm{i} \cdot p \cdot \frac{\mathrm{e}^{\mathrm{i}p|\boldsymbol{x}-\boldsymbol{x}'|}}{2\,p} = -\frac{\mathrm{e}^{\mathrm{i}p|\boldsymbol{x}-\boldsymbol{x}'|}}{4\,\pi\,|\boldsymbol{x}-\boldsymbol{x}'|} = \frac{1}{2\,m}\langle\boldsymbol{x}|\frac{1}{E - H_0 + \mathrm{i}\varepsilon}|\boldsymbol{x}'\rangle$$

[For those of you who, like me, have forgotten the details of the residue theorem, here it is: "Suppose that f is analytic on a simply-connected domain D except for a finite number of isolated singularities at points z_1, ..., z_N of D. Let γ be a piecewise smooth positively oriented simple closed curve in D that does not pass through any of the points $z_1, \ldots, z_n$. Then

$$\int_\gamma f(z)\,\mathrm{d}z = 2\pi\,\mathrm{i} \sum_{z_k \text{ inside } \gamma} \mathrm{Res}(f; z_k)$$

where the sum is taken over all those singularities z_k of f that lie inside γ." (Stephen D. Fisher, *Complex Variables*, second edition, Dover 1999.) And how to calculate residues: "Suppose that F and G are analytic functions on the disc $\{z\colon |z - z_0| < r_0\}$ with $G(z_0) = 0$ but $G'(z_0) \neq 0$. Show that $\mathrm{Res}(F/G; z_0) = F(z_0)/G'(z_0)$" (The same book, example 4, chapter 2.) Here we would have $z_0 = p + \mathrm{i}\varepsilon$ and $G(q) = q^2 - p^2 - \mathrm{i}\varepsilon \Rightarrow G'(q) = 2\,q$. We must have taken the limit $\varepsilon \to 0$ immediately after we applied the residue theorem, so that $G'(z_0) \to G'(p) = 2p$.]

Now we have the Lippmann-Schwinger equation in a very explicit form.

$$\psi(\boldsymbol{x}) = \frac{1}{(2\,\pi)^{3/2}}\,\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} - 2\,m \int \mathrm{d}\boldsymbol{x}'\,\frac{\mathrm{e}^{\mathrm{i}p|\boldsymbol{x}-\boldsymbol{x}'|}}{4\,\pi\,|\boldsymbol{x}-\boldsymbol{x}'|}\,V(\boldsymbol{x})\,\psi(\boldsymbol{x})$$

This is exact, but useless. One more step with this, to get to the cross section.

We had some $V(\boldsymbol{x}')$, and we are interested in the behaviour of $\psi(\boldsymbol{x})$ very far away. If this were a nuclear physics experiment $V \neq 0$ for $|\boldsymbol{x}'| \sim$ a few femtometres. In an atomic physics experiment, we would have $\boldsymbol{x}'$ on the order of Ångströms. So we make the approximation $V \neq 0$ for $|\boldsymbol{x}'| \ll |\boldsymbol{x}|$, as the detector would be $|\boldsymbol{x}| \sim$ a few metres away. We shall continue tomorrow.